Chapter 1 – Trigonometric Functions

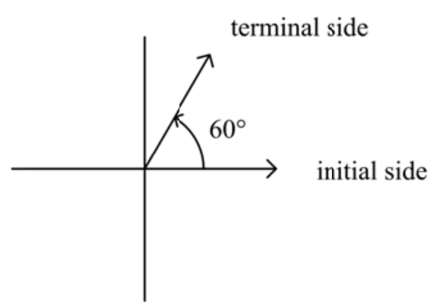
Section 1.1 – Measuring Angles

The Big Picture

Trigonometry revolves around angles so we're going to start our discussion by looking at different ways that angles are measured. In the process, we'll add a few new definitions and make some connections back to ideas from geometry.

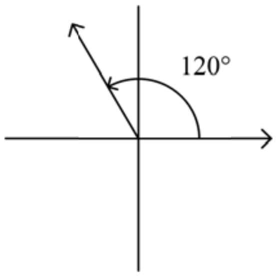
Decimal Degrees

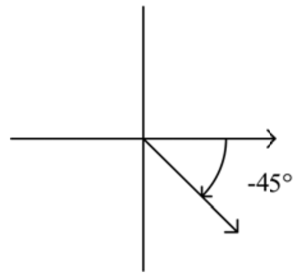
Decimal degrees are the same degrees that you learned about in geometry: a circle is divided into 360 equal parts, each of which is called a degree. We're going to add a little twist to that by taking geometry's angles and putting them on algebra's coordinate plane. You can see the standard way of doing this in the diagram to the right. The vertex of the angle is at the origin and one side of the angle, called the **initial side**, runs along the positive *x*-axis. When the angle is drawn this way, the side that isn't on the *x*-axis is called the **terminal side** and the angle itself is in the **standard position**.



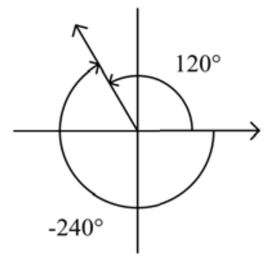
In geometry classes, angles are usually limited to being between 0° and 180°. For our purposes, there aren't going to be any limits on the angle's size. For example, the diagram on the left on the next page shows a 120° angle drawn in the standard position.

Another twist that we're going to put on geometry angles is that our angles can have a negative measure. When you're looking at the graph, a negative angle is drawn with the terminal side going clockwise where a positive angle is drawn going counter-clockwise.

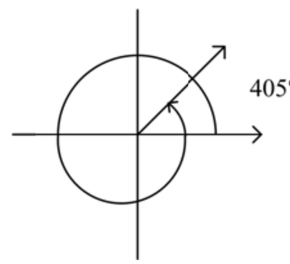




This is similar to what you saw back in Algebra 1 with number lines. If you started at 0 and went to the right, you would end up on a positive value where, if you went to the left, you would end up on a negative value.



The diagram to the right shows an interesting consequence of letting angles be negative. If you look at the terminal side going into the second quadrant, there are two ways to get to it from the initial side. If we go counter-clockwise around the circle, we get a 120° angle; if we go clockwise, we get a -240° angle. Two angles like this pair that are in standard position and have the same terminal side but different measures are called **coterminal angles**.

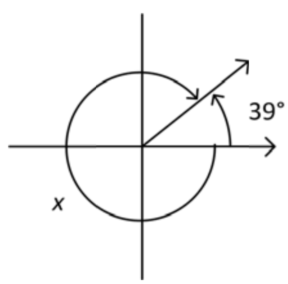


We can even have angles greater than 360°. For example, if I wanted to graph a 405° angle, I would think of 405° as being 360° + 45°. That tells me that I can get the graph by starting in the standard position, going all the way around the circle once (360°) and then going 45° further to get the final result. In other words, a 405° angle is visually the same as a 45° angle.

Example 1 – Coterminal Angles

If an angle measures 39°, what's the measure of a coterminal angle?

The diagram below shows a 39° angle and its coterminal angle.



The two angles, together, make up an entire circle so their measures have to add up to 360°:

$$x + 39° = 360°$$

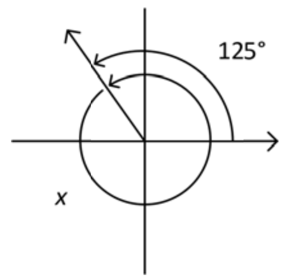
$$x = 321°$$

Because the coterminal angle has to go clockwise from the initial side, its measure has to be negative so the final value is -321°.

Example 2 – Coterminal Angles

Give the measure of two angles that are coterminal with a 125° angle.

The diagram below shows the original 125° angle. If we start at the terminal side of that angle and go another full loop around the circle, we'll come back to the same terminal position. The measure of that angle would be



$$(\text{new angle}) = (\text{original } 125°) + (\text{additional full circle})$$

$$(\text{new angle}) = 125° + 360°$$

$$(\text{new angle}) = 485°$$

To get the second angle, we can do the same thing but going the other way (clockwise) around the circle.

$$(\text{new angle}) = (\text{original } 125°) + (\text{additional full reverse circle})$$

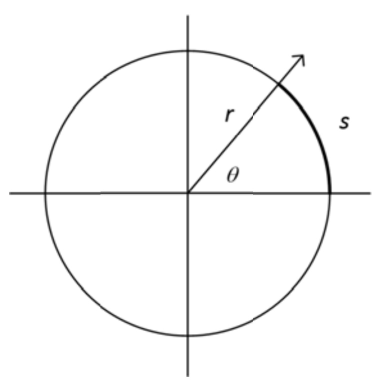
$$(\text{new angle}) = 125° + (-360°)$$

$$(\text{new angle}) = -235°$$

Radians

Decimal degrees work for a lot of things but they have some disadvantages. For example, it take a lot of degrees (360 of them) to get to a full circle. That's going to be a big disadvantage when we talk about graphing in Chapter 3. The size of a whole circle, 360°, is also a pretty arbitrary number. The radian system addresses both of these issues. To see how radians are defined take a look at the diagram below. It starts with a circle whose radius is *r* and an angle whose measure we'll call θ. Notice how the angle cuts off a section of the circle. If the length of that arc is *s* then we'll define the radian measure of the angle as

$$\theta = \frac{s}{r}$$



There are some potential issues with that definition but, before we look at those, let's see what some actual angles look like.

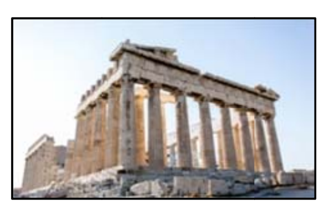
Suppose we start with a circle whose radius is 1, i.e. $r = 1$. That simplifies our definition to

$$\theta = s$$

A Quick Introduction to the Greek Alphabet

In trigonometry, we usually label angles using Greek letters. Here's a quick list of the ones that we'll be using:

alpha	α
beta	β
gamma	γ
theta	θ
phi	ϕ



So suppose we have an angle that equals half of the circle. The length of that arc is half the circumference of the circle or

$$\theta = s = \frac{C}{2} = \frac{2\pi r}{2} = \frac{2\pi \cdot 1}{2} = \frac{2\pi}{2} = \pi$$

So the radian measure of a half of a circle is π.

Hopefully, you can see pretty quickly from there that the radian measure of a whole circle is 2π because the circumference of the whole circle is twice the circumference of the semi-circle.

There's a big question that I kind of ignored in the previous calculations: Suppose we picked a different size circle, i.e. a circle with a radius other than 1. Will we get the same result? If we don't then we have a problem because we don't want to have multiple sizes for the same angle.

Question: Does the radian measure of an angle stay the same regardless of the size of the circle?

To see what's going on here, we need to go back to our original definition:

$$\theta = \frac{s}{r}$$

Now suppose we look at the same angle but with a circle whose radius is twice that of the original circle, i.e. a circle whose radius is $2r$. What will be the value of θ for that circle?

First, let θ' be the measure of the angle in the new circle.[1] What we need to show is that $\theta = \theta'$, i.e. that the angle measures are the same in both circles. To do that, we need to start by looking at the values of the radius and arc length in the new circle. We already decided that the new radius is twice the old one so, if we call the new radius r', we have

$$r' = 2r$$

If we double the radius then we also double the circumference and the portion of the circumference that's intercepted by our angle, i.e. if s' is the length of the new arc then

$$s' = 2s$$

Now we're ready to pull out our radian formula:

$$\theta' = \frac{s'}{r'}$$

If we substitute $2r$ for r' and $2s$ for s', the equation becomes

$$\theta' = \frac{2s}{2r} = \frac{s}{r} = \theta$$

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[1] Did you notice the "apostrophe" after the θ? That's a notation that we sometimes use to indicate a second value for something. In this case, θ is the measure of the angle in the first circle and θ' is the measure of the angle in the second circle.

Which is what we wanted to show, i.e. that the measure of the angle is the same for the original circle (θ) and the new circle (θ').

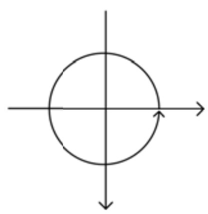
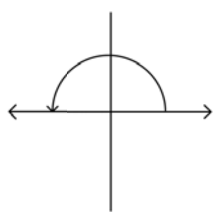
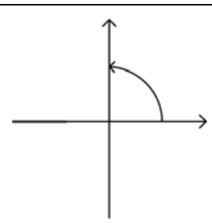
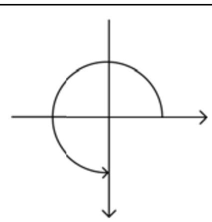
In this argument, I showed that the measure stays the same if the size of the circle is doubled. In the Exercises, you'll work through an argument that shows that the radian measure stays the same for any size circle.

Some Values

Now that we know that the size of the circle doesn't affect the measure of the angle, we can find some specific values by looking at a circle whose radius is 1, called a **unit circle**. For this circle, our radian measure definition simplifies to

$$\theta = \frac{s}{r} = \frac{s}{1} = s$$

The table below gives the radian measures for several common angles. The circumference of the whole circle is $C = 2\pi r = 2\pi \cdot 1 = 2\pi$ so all of our radian values will be a fraction of that length.

Angle	Description	Radians	Degrees
	a full circle	$\theta = s = C = 2\pi$	360°
	half of a circle	$\theta = s = \frac{C}{2} = \frac{2\pi}{2} = \pi$	180°
	quarter of a circle	$\theta = s = \frac{C}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$	90°
	three quarters of a circle	$\theta = s = \frac{3C}{4} = \frac{3 \cdot 2\pi}{4} = \frac{3\pi}{2}$	270°

If you look carefully at the last two columns in the table, you'll see that I used the traditional "small circle", °, for degrees but I didn't put anything after the radians. Sometimes you'll see "rad" used to indicate a value is in radians, e.g. π rad. However, it's also common to put no units after a radian measure, e.g. just π or π / 3. As a general rule, if there's no unit after an angle measure, you should assume that the value is in radians.

Converting Radians

Now that we have our new angle measure and some idea of what its values look like, we need a way to convert back and forth between degrees and radians. The "half circle" value gives us an easy conversion factor that we can use

$$\frac{180°}{\pi \text{ rad}} \quad \text{or} \quad \frac{\pi \text{ rad}}{180°}$$

Example 3 – Radian Conversions

Convert 45° into radians.

For this conversion, I'll use the $\frac{\pi \text{ rad}}{180°}$ conversion factor.

$$45° = 45° \cdot \frac{\pi \text{ rad}}{180°} = \frac{\pi}{4} \text{ rad}$$

Notice how I left π in the result. That's not unusual when we're working with radians. They're often given as a fraction of π

Example 4 – Radian Conversions

Convert 2.6 rad to degrees.

For this conversion, I'll use the other conversion ratio: $\frac{180°}{\pi \text{ rad}}$.

$$2.6 \text{ rad} = 2.6 \text{ rad} \cdot \frac{180°}{\pi \text{ rad}} = \frac{468}{\pi}° = 148.97°$$

In this case, I didn't leave π in the answer because degree measures are always given as decimal numbers.

Example 5 – Radian Conversions

Convert 143° into radians. Give the result to two decimal places.

For this conversion, I'll use the $\frac{\pi \text{ rad}}{180°}$ conversion factor again.

$$143° = 143° \cdot \frac{\pi \text{ rad}}{180°} = \frac{143\pi}{180} \text{ rad} \approx 2.50$$

For this result, I replaced the π with 3.1415926 and simplified the expression to a decimal because the question explicitly asked for the answer in that format.

Example 6 – Radian Conversions

Convert $\frac{4\pi}{6}$ to degrees.

For this calculation, we could use the conversion factor but there's a quick shorthand that I use when the radian value has π in it: just replace the π with 180. That gives us

$$\frac{4\pi}{6} \text{ rad} = \frac{4 \cdot 180°}{6} = 120°$$

When you use the shortcut in Example 6, it's important that you only do it when the π is in the numerator of the fraction. It works because, when you multiply by the 180 / π conversion factor, the π in the numerator of the angle measure cancels with the π in the denominator of the conversion factor. If you have a π in the denominator of the angle measure, e.g. something like $\frac{3}{\pi}$, then you would have to multiply by the conversion factor like I did in Example 4.

Exercises

Convert the following degree measures into radian measures. Give the exact value using π.

1. 135°
2. 90°
3. 120°
4. 60°
5. -240°
6. -92°
7. 48°
8. 212°

Convert the following radian measures into degree measures.

9. $\frac{3\pi}{4}$
10. $\frac{11\pi}{7}$
11. $-\frac{7\pi}{4}$
12. $\frac{5\pi}{6}$
13. 3.1411
14. $\frac{16\pi}{9}$
15. -6.11
16. $\frac{7\pi}{12}$

Graph the given angles in standard position. Give the measures of two coterminal angles for each angle, one of which is positive and the other negative.

17. 150°
18. 215°
19. -15°
20. $\frac{3\pi}{4}$
21. 42°
22. $\frac{9\pi}{4}$
23. -2.5
24. $\frac{8\pi}{15}$
25. $\frac{7\pi}{2}$
26. 72°
27. $-\frac{7\pi}{6}$
28. 4.18

Analysis

29. In the discussion of radians, I showed that if an angle measures θ radians in a circle whose radius is *r* then it also measures θ radians in a circle whose radius is 2*r*. To be sure that an angle has exactly one radian value, we need to show that we get the same value for *every* circle. Write an argument that does this by letting *C* be a circle whose radius is *kr* where *k* is some real number. Show that if angle measures θ radians for a circle whose radius is *r* then it also measures θ radians for a circle whose radius is *kr*.

Technical Writing

30. Referring back to the section on decimal degrees, list all of the consequences that resulted from not limiting angles to being between 0° and 90°.
31. Explain the difference between the terminal and initial sides of an angle.

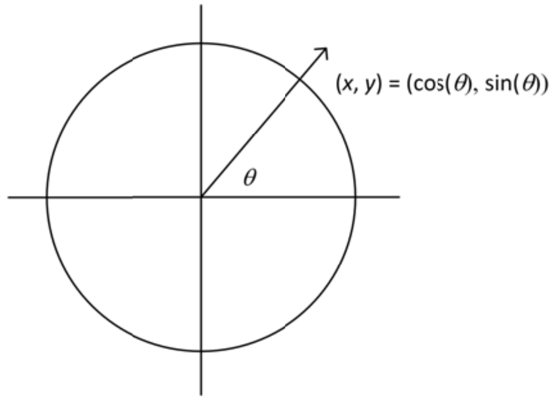
Section 1.2 – The Two Core Functions

The Big Picture

In your high school geometry class, you may have learned some trigonometry that was based on the sides of triangles. We're going to take a different approach that, as you'll see in the next section, gives us a lot more flexibility. In this section, we're going to develop the two core trigonometry functions, sine and cosine, and look at some of their specific values.

The Sine and Cosine

Suppose we start with a unit circle, i.e. a circle whose radius is 1, and we draw an angle in the standard position. The coordinates of the point where the line crosses the circle give us the cosine and sine of the angle. The *x*-coordinate is the cosine value and the *y*-coordinate is the sine value.

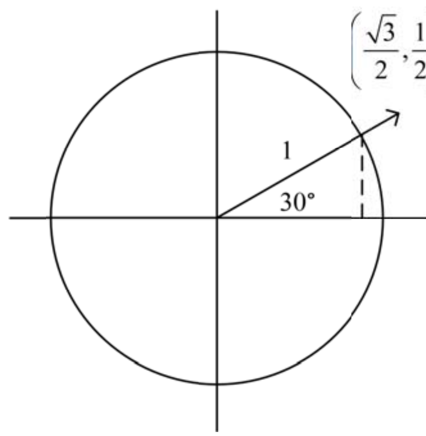


This is the fundamental concept that all of trigonometry is based on so before we move on to looking at some specific values, I want to make sure it's clear. If you start with an angle, θ, and you calculate the coordinates of the point, (x, y), where it crosses a unit circle, then $\cos(\theta) = x$ and $\sin(\theta) = y$.

Notice how I abbreviated the cosine and sine functions as "cos" and "sin". Besides being a little easier than writing out "cosine" and "sine", writing it this ways helps to emphasize that the sine and cosine are functions in the sense that they take an angle and turn it into a number. When you're reading something like $\sin(\theta)$ or $\cos(\theta)$, you should do it just like you would any other function like $f(x)$, i.e. as "sine of θ" and "cosine of θ".

Some Specific Values

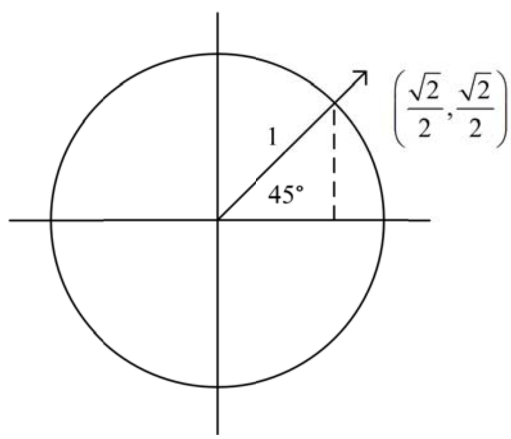
Now that we have our definition, let's see how this applies in practice by finding the sine and cosine of some specific angles.



Suppose that we wanted to find cos(30°) and sin(30°), i.e. the sine and cosine of 30°. I've drawn this angle in a unit circle in the diagram to the right. The key part to finding these values is the dotted line that I added that's perpendicular to the *x*-axis. Adding that line gives us a 30-60-90 triangle whose hypotenuse is 1. In that special triangle, the side opposite the 30° angle is 1/2 and the side adjacent to the 30° angle is $\sqrt{3}/2$. In other words, to get to the point where the angle crosses the circle, we have to go over $\sqrt{3}/2$ and up 1/2 so its coordinates are ($\sqrt{3}$ /2, 1/2).

Now think back to our sine and cosine definition. The sine of 30° is the *y*-coordinate of the point which means that sin(30°) = 1/2. On the other hand, the cosine of 30° is the *x*-coordinate of the point which means that cos(30°) = $\sqrt{3}$ /2.

Let's look at another example. Say we want to find sin(45°) and cos(45°). We can use the same approach to do this as we did with a 30° angle. First, I drew the 45° angle inside a unit circle and drew a perpendicular segment from the point where the circle and the line cross down to the *x*-axis, making a right triangle. This is a 45-45-90 triangle so, because the hypotenuse of the triangle is 1, the lengths of the two sides must both be $\sqrt{2}$ /2. That makes the coordinates of the intersection ($\sqrt{2}$ /2, $\sqrt{2}$ /2). The *x*-coordinate of that point is the cosine of our angle and the *y*-coordinate is the sine which means we have

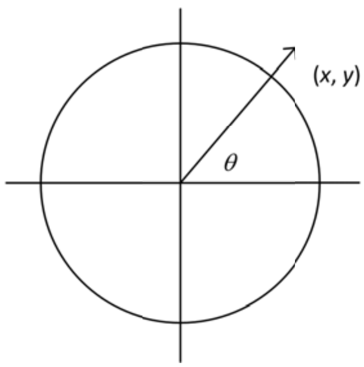


$$\sin(45°) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos(45°) = \frac{\sqrt{2}}{2}$$

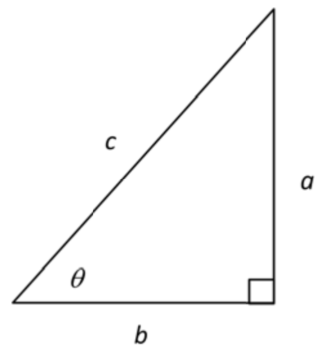
Looking Back to Triangles

Question: Does the geometry definition of the sine of an angle give the same values as the unit circle definition?

In your geometry class, you probably saw the sine and cosine function defined in terms of right triangles. For example, the sine of an angle was defined the length of the side opposite that angle divided by the length of the hypotenuse. In this section, I'm going to go through an argument showing that the geometry, right triangle-based definitions of the sine function give us the same results as we get from using our new and improved unit circle-based definition. In other words, if we have these two scenarios:



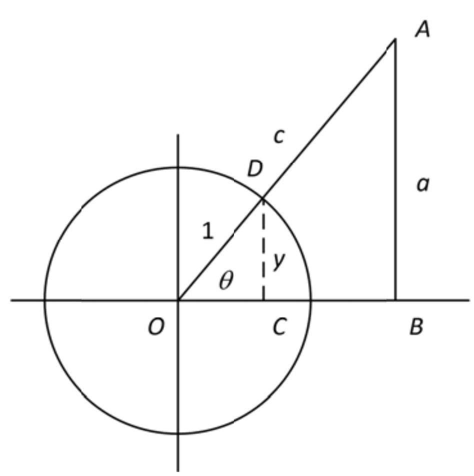
$\sin(\theta) = y$



$\sin(\theta) = \frac{a}{c}$

We get the same value for $\sin(\theta)$ for each one, i.e. we need to show that $y = \frac{a}{c}$.

To do that, I'm going to use the diagram to the right which looks pretty messy so let me explain it one step at a time. The first thing I did is draw our triangle over the circle. That's ΔAOB with its angle θ overlapping the angle in the circle. Next I drew the perpendicular line from D down to the x-axis, just like I did in the previous problems in this section. The coordinates of D are (x, y) which means that $DC = y$; O is a unit circle which means that the radius $\overline{DO}$ has length 1.



Now, take a look at the two triangles, ΔABO and ΔDCO. $\angle B$ and $\angle C$ are both right angles so $\angle B \cong \angle C$. $\angle O$ is the same in both triangles so $\angle O \cong \angle O$ which means that the two triangles are similar by the AA Similarity Theorem: $\Delta ABO \sim \Delta DCO$.

Here comes the key part of the argument. Because the two triangles are similar, their corresponding sides are proportional. In particular:

$$\frac{DC}{OD} = \frac{AB}{OA}$$

Looking back at the diagram, $OD = 1$ because the circle is a unit circle; $DC = y$ in the smaller triangle; $AB = a$ in the larger triangle and $OA = c$

in the larger triangle. If we substitute those values into the proportion, it becomes

$$\frac{y}{1} = \frac{a}{c}$$
$$y = \frac{a}{c}$$

But, if you look back at the beginning of this discussion, that's what we wanted to prove: The sine value that we get from the circle, *y*, is equal to the sine value that we get from the triangle, *a* / *c*.

Before we close this section, I want to take a step back and look at what we've covered because we've hit on some pretty important topics.

1. We came up with a new definition for the sine and cosine using a circle instead of a triangle. In the next section, we'll see why this definition lets us do a lot more than the triangle version.

2. We showed that the right triangle-based "opposite over hypotenuse" definition gives us the same results for the sine of an angle as our unit-circle based "*y*-coordinate" definition does.

Analysis

1. What are cos(60°) and sin(60°)?
2. By looking at the coordinates of points on the graph, find cos(0°), sin(0°), cos(90°) and sin(90°).
3. Suppose you have a circle of radius *r* and an angle drawn in that circle in the standard position. What are the coordinates of the point where the angle's terminal side crosses the circle?
4. The right-triangle based definition of the cosine of an angle says that the cosine is the length of the side adjacent to the angle divided by the hypotenuse. Referring back to the triangle we used at the beginning of the "Looking Back at Triangles" section, it would be cos(θ) = *b* / *c*. Using the argument in that section as a model, show that that triangle-based cosine definition gets the same results as our "*x*-coordinate" definition using a unit circle.

Technical Writing

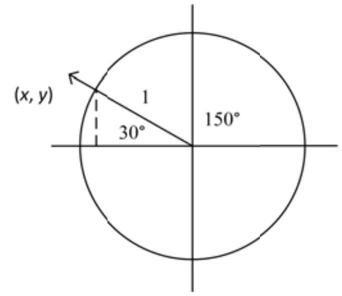
5. Explain the definition of the sine and cosine functions in your own words.

Section 1.3 – Values of the Sine and Cosine

The Big Picture

Now that we have definitions for our two main functions, we're going to look at how to find the functions' values for specific angles. We'll have exact values for a group of "special" angles; for all of the others, we'll have to fall back on using a calculator.

Some Special Angles



Suppose we wanted to find cos(150°). Using our definition from the previous section, we need to find the coordinates of the point where a 150° angle crosses the unit circle. At first glance, this may seem like a problem. There is no triangle with a 150° angle where we know anything about the side lengths. However, suppose we draw the dashed line in the diagram. That makes a 30-60-90 triangle that we can use to get the coordinates of the point where the terminal side of the 150° angle crosses the circle. Because the hypotenuse (the radius of the circle) is 1, the coordinates must be the same as they were for a 30° but with a negative *x*-coordinate: $(-\sqrt{3}/3, 1/2)$. That tells us that

$$\cos(150°) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(150°) = \frac{1}{2}$$

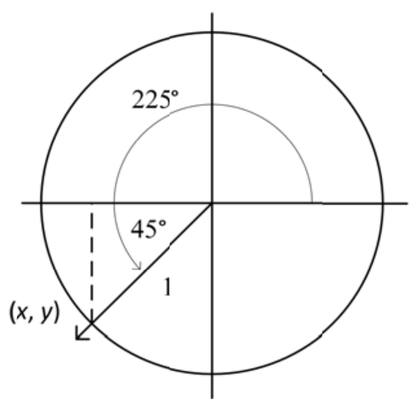
The 30° that we drew in the diagram is called the **reference angle** of the original 150° angle. Using reference angles we can get the sine and cosine of any angle by finding the sine or cosine of its reference angle and then adjusting the sign based on the quadrant the angle is in.

Example 1 – Trigonometric Values

Find sin(225°) and cos(225°).

The diagram below has the 225° and its reference angle. Because a semi-circle is 180°, the reference angle has to be

$$225° - 180° = 45°$$



The lengths of the sides of the triangle are both $\sqrt{2}/2$ which means that the coordinates of the point are

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

which makes the trigonometric values

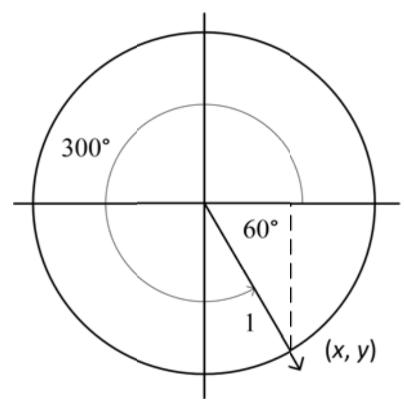
$$\cos(225°) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(225°) = -\frac{\sqrt{2}}{2}$$

Example 2 – Trigonometric Values

Find sin(300°) and cos(300°).

The diagram below has the 300° and its reference angle. Because the whole circle is 360°, the reference angle has to be

$$360° - 300° = 60°$$



The lengths of the sides of the triangle are $\sqrt{3}/2$ and 1/2 which means that the coordinates of the point are

$$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

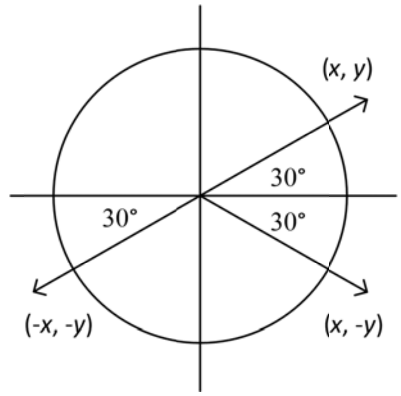
which makes the trigonometric values

$$\cos(300°) = \frac{1}{2} \quad \text{and} \quad \sin(300°) = -\frac{\sqrt{3}}{2}$$

Example 3– Trigonometric Values

For what angles is the sine of the angle -1/2?

We've already seen that sin(30°) = +1/2. So for the sine to be -1/2, we need two angles that have a 30° reference angle but that are in the third and fourth quadrants because those are the quadrants where the *y*-coordinates of the points are negative.



The angle in the third quadrant with a 30° reference angle is

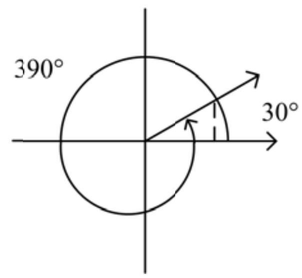
$$180° + 30° = 210°$$

The angle in the fourth quadrant is

$$360° - 30° = 330°$$

That tells us that sin(210°) = -1/2 and sin(330°) = -1/2.

Periodicity



Suppose I asked you to find cos(390°). Because

$$390° = 360° + 30°$$

the terminal side of our 390° angle is in the same position as the terminal side of a 30° angle. That means that

$$\cos(390°) = \cos(360° + 30°) = \cos(30°) = \frac{\sqrt{3}}{2}$$

Now suppose I asked you for cos(750°). This angle also has the same terminal side as a 30° angle except that this time we have to make two revolutions around the circle:

$$750° = 720° + 30° = 2 \cdot 360° + 30°$$

so

$$\cos(750°) = \cos(2 \cdot 360° + 30°) = \cos(30°) = \frac{\sqrt{3}}{2}$$

If you went another revolution around the circle you'd get the same result again:

$$\cos(n \cdot 360° + 30°) = \cos(30°) \text{ where } n \text{ is any integer}$$

In other words, if you take an angle and add a multiple of 360° to it then you get the same cosine (and sine) values. This behavior where the value of a function repeats at regular intervals is called **periodicity**. The interval at which the function starts repeating is called its **period**. For our sine and cosine functions the period is 360°.

Periodicity	For any integer *n* $\cos(360° \cdot n + x) = \cos(x)$ $\sin(360° \cdot n + x) = \sin(x)$

Example 4 – Trigonometric Values

Find cos(390°).

When the angle measure is greater than 360°, you can apply the periodicity rule by subtracting 360° until you get to an angle between 0° and 360°.

$$390° - 360° = 30°$$

That means that

$$390° = 360° \cdot 1 + 30°$$

Using the periodicity formula, we have

$$\cos(390°) = \cos(360° \cdot 1 + 30°) = \cos(30°) = \frac{\sqrt{3}}{2}$$

Example 5 – Trigonometric Values

Find sin(-480°).

When the degree measure is less than 0°, you can use the same approach but adding 360° instead of subtracting.

$$-480° + 360° = -120°$$

$$-120° + 360° = 240°$$

I had to add two 360°'s to the original -480° which means that

$$-480° = 240° + 360° \cdot (-2)$$

The periodicity formula tells us that

$\sin(-480°) = \sin(240° + 360° \cdot (-2)) = \sin(240°) =$
$-\frac{\sqrt{3}}{2}$

Example 6 – Trigonometric Values

Give three angles whose sine is $\frac{\sqrt{2}}{2}$.

We've already seen that $\sin(45°) = \frac{\sqrt{2}}{2}$. The periodicity rule, in practical terms, says that we can start at 45° and get the same sine values by adding multiples of 360° to it.

$$45° + 360° \cdot 1 = 405°$$

$$45° + 360° \cdot 2 = 765°$$

$$45° + 360° \cdot 3 = 1125°$$

According to the periodicity rule sin(405°), sin(756°) and sin(1125°) must also equal $\frac{\sqrt{2}}{2}$.

Example 7 – Trigonometric Values

Give three negative angles whose cosine is $\frac{1}{2}$.

We've already seen that $\cos(60°) = \frac{1}{2}$. The periodicity rule works for negative multiples of 360° so we can get negative angles by multiplying by -1, -2 and -3.

$$60° + 360° \cdot (-1) = -300°$$

$$60° + 360° \cdot (-2) = -660°$$

$$60° + 360° \cdot (-3) = -1020°$$

According to the periodicity rule cos(-300°), cos(-660°) and cos(-1020°) must also equal $\frac{1}{2}$.

Non-Special Values

At this point, we've seen how to calculate the trigonometric functions for some specific angles and these are definitely values that you need to know but what about all the other angles? For example, suppose you wanted to know the value of cos(28°) or sin(-112°), i.e. angles that don't correspond to one of our special triangles? In the next chapter, we'll see some formulas that will let us expand the angles for which we can get exact values but, for most angles, you're going to have to fall back on using your calculator. When you do that, you need to be sure to double check the calculator's angle measure setting. Any calculator with trigonometric functions on it is going to accept the angle measures in both degrees and radians. If you have it set for one and enter your angle measures in the other, you're going to get incorrect results.

Since we're talking about using a calculator, there's some terminology you should be aware of. If a question asks for an "exact answer", that means you should use the special angles that we've talked about in this section and the previous one to get an answer that's either a fraction like 1/2 or that has a square root in it like $\sqrt{3}$ / 2. If the question asks for an approximate value then you can use your calculator to get a decimal result that you'll probably have to round.

Exercises

Fill in the values in the following table. In the "*x* rad" column, put the radian version of the angle measure.

	$x°$	x rad	$\cos(x)$	$\sin(x)$
1.	0°		1	
2.	30°			$\frac{1}{2}$
3.	45°		$\frac{\sqrt{2}}{2}$	
4.	60°	$\frac{2\pi}{3}$		
5.	90°		0	
6.	120°			
7.	135°			
8.	150°		$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
9.	180°	π		

	$x°$	x rad	$\cos(x)$	$\sin(x)$
10.	210°			
11.	225°		$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
12.	240°			
13.	270°			
14.	300°		$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
15.	315°	$\frac{7\pi}{4}$		
16.	330°			
17.	360°			

Find the following values.

18. cos(-60°)
19. sin(-225°)
20. cos(540°)
21. cos(480°)
22. sin(960°)
23. sin(-420°)
24. cos(585°)
25. sin(-495°)

Use a calculator to find the following values to three decimal places. Be sure to distinguish between degrees and radians in the angle measures.

26. cos(42°)
27. sin(76°)
28. cos(302°)
29. cos(2.11)
30. sin(7.11)
31. sin(-12°)
32. cos(103°)
33. sin(.658)

34. Give three angles whose cosine is 0.

35. Give three angles whose sine is $\frac{\sqrt{3}}{2}$.

36. Give three negative angles whose cosine is $-\frac{1}{2}$.

37. Give three negative angles whose sine is -1.

Technical Writing

30. Explain what it means for the sine function to be periodic and why it has that property.

Section 1.4 – The Other Trigonometric Functions

The Big Picture

The sine and cosine are the two core trigonometric functions but there are four others that you need to know. All four are based on the values of the sine and cosine and, technically, you can do almost everything you can do with these functions with just the sine and cosine. However, there will be situations where using these "derived functions" will get you to an answer faster than using the two "core functions".

The Four Derived Functions

The following table summarizes the new functions and their definitions. The values in the Unit Circle Definition give the functions' definitions in terms of the coordinates where the angle crosses the unit circle. (Remember that if the angle's terminal side crosses the unit circle at (x, y) then $\cos(\theta) = x$ and $\sin(\theta) = y$.)

Name	Function	Function Definition	Unit Circle Definition
tangent	$\tan(\theta)$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\tan(\theta) = \frac{y}{x}$
secant	$\sec(\theta)$	$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\sec(\theta) = \frac{1}{x}$
cosecant	$\csc(\theta)$	$\csc(\theta) = \frac{1}{\sin(\theta)}$	$\csc(\theta) = \frac{1}{y}$
cotangent	$\cot(\theta)$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$	$\cot(\theta) = \frac{x}{y}$

Some Specific Values

Now that we have those definitions, let's do the same thing we did with the sine and cosine functions and look at some specific values of each function.

Example 1 – Trigonometric Values

Find tan(30°).

The definition of the tangent function says that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

or, substituting 30° for θ,

$$\tan(30°) = \frac{\sin(30°)}{\cos(30°)}$$

If we substitute the values we already know for sin(30°) and cos(30°), we get

$$\tan(30°) = \frac{\sin(30°)}{\cos(30°)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

so $\tan(30°) = \frac{1}{\sqrt{3}}$.

Example 2 – Trigonometric Values

Find csc(π).

The definition of the cosecant function tells us that

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

so

$$\csc(\pi) = \frac{1}{\sin(\pi)} = \frac{1}{0}$$

which is a problem because 1/0 is undefined. This can happen with all of our new functions because their denominators will all be 0 for some angles. Our final answer here would be that csc(π) is undefined.

Periodicity

Our new functions all have periodic behavior that's similar to the two base functions. With the secant and cosecant functions that isn't much of a surprise, e.g. if cos(θ) repeats every 2π radians then it isn't a big stretch to conclude that 1 / cos(θ) is also going to repeat with the same frequency. With the tangent and cotangent functions, it gets a little trickier. Those functions also behave periodically but, unlike the other four functions, they repeat every π radians.

Periodicity	For any integer n $\sec(x + 2\pi n) = \sec(x)$ $\csc(x + 2\pi n) = \csc(x)$ $\tan(x + \pi n) = \tan(x)$ $\cot(x + \pi n) = \cot(x)$

This time, I wrote the formulas out in radians, i.e. using 2π and π as the multiples. You'll get the same results if you replace the 2π's with 360° and the π's with 180°.

Example 4 – Trigonometric Values

Find $\sec\left(\frac{13\pi}{6}\right)$**.**

When you're answering these kinds of questions, it helps to remember that 2π is equal to 12π / 6. Thinking of it that way, it's easy to see that 13π / 6 is

Example 5 – Trigonometric Values

Find cot(-480°).

The quickest, most practical way to answer this kind of question is to start with -480° and keep adding 180° to it until you get a value between 0° and 360°. (We're adding 180° because that's the period of the greater than 2π so we can simplify the expression by subtracting 2π from it:

$$\frac{13\pi}{6} - 2\pi = \frac{13\pi}{6} - \frac{12\pi}{6} = \frac{\pi}{6}$$

or

$$\frac{13\pi}{6} = \frac{\pi}{6} + 2\pi$$

That tells us that

$$\sec\left(\frac{13\pi}{6}\right) = \sec\left(\frac{\pi}{6} + 2\pi\right) = \sec\left(\frac{\pi}{6}\right)$$

Now we can use the function definition of the secant function to get its value:

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Remember that when you're working with square roots, you should always eliminate square roots from the denominator of your answer. In this case, I can do that by multiplying the top and bottom by $\sqrt{3}$.

$$\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

cotangent function.)

-480° + 180° = -300°

-300° + 180° = -120°

-120° + 180° = 60°

This means that

cot(-480°) = cot(60°)

Now, we can apply the function definition to get a numeric value.

$$\cot(-480°) = \cot(60°)$$
$$\cot(-480°) = \frac{\cos(60°)}{\sin(60°)}$$
$$\cot(-480°) = \frac{1/2}{\sqrt{3}/2}$$
$$\cot(-480°) = \frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$
$$\cot(-480°) = \frac{1}{\sqrt{3}}$$

This is another situation where we need to eliminate the square root in the denominator by multiplying the top and bottom by $\sqrt{3}$.

$$\cot(-480°) = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$\cot(-480°) = \frac{\sqrt{3}}{3}$$

Example 6 – Trigonometric Values

Give three angles whose secant is $\sqrt{2}$.

To answer a question like this, it's easiest to convert the secant value to a cosine value and then use what we know about the cosine function to get the angles. To do that, I'll use the formula

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

We're looking for the angles where $\sec(\theta) = \sqrt{2}$ so, if I replace the sec(θ) with $\sqrt{2}$, the equation becomes

$$\sqrt{2} = \frac{1}{\cos(\theta)}$$
$$\sqrt{2}\cos(\theta) = 1$$
$$\cos(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We know that that's true for $\theta = 45°$ and we can get two more values by adding multiples of 360° to 45°:

$$45° + 360° \cdot 1 = 405°$$

$$45° + 360° \cdot 2 = 765°$$

So the cosine function equals $\sqrt{2}/2$ when $\theta = 45°$, 405° and 765° which means the secant function equals $\sqrt{2}$ for those angles.

Example 7 – Trigonometric Values

Give three negative angles whose tangent is $\sqrt{3}$.

We can use the same approach with the tangent function as we did with the secant but we're going to have to be a little more creative about it. We'll start with the tangent formula

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

We're looking for angles where $\tan(\theta) = \sqrt{3}$ or

$$\sqrt{3} = \frac{\sin(\theta)}{\cos(\theta)}$$

What makes this a little tricky is that we have two functions on the right side. To get this into a form that I can work with, I'm going to think of the $\sqrt{3}$ as $\sqrt{3}/1$.

$$\frac{\sqrt{3}}{1} = \frac{\sin(\theta)}{\cos(\theta)}$$

Now, I'm going to divide the top and bottom of the left hand side by 2.

$$\frac{\sqrt{3}/2}{1/2} = \frac{\sin(\theta)}{\cos(\theta)}$$

That equation is in a form I can work with. We're looking for an angle where $\sin(\theta) = \sqrt{3}/2$ and $\cos(\theta) = 1/2$.

$\theta = 60°$ gives us both of those values so $\tan(60°) = \sqrt{3}$. We're looking for negative values so we can get the results that we need by subtracting multiples of 180° from 60°.

$$60° - 1 \cdot 180° = -120°$$

$$60° - 2 \cdot 180° = -300°$$

$$60° - 3 \cdot 180° = -480°$$

So the tangent function is $\sqrt{3}$ when $\theta = -120°$, -300° and -480°.

Exercises

Fill in the values in the following table.

	$x°$	tan($x°$)	sec($x°$)	csc($x°$)	cot($x°$)
1.	0°				
2.	45°		$\sqrt{2}$		
3.	60°			$\frac{2\sqrt{3}}{3}$	
4.	90°				
5.	120°				
6.	180°				

	$x°$	tan($x°$)	sec($x°$)	csc($x°$)	cot($x°$)
7.	210°		$-\frac{2\sqrt{3}}{3}$		
8.	240°				
9.	270°				
10.	315°				
11.	330°			-2	
12.	360°				

Find the following values.

18. cot(-60°) 19. tan(-225°) 20. csc(540°) 21. sec(480°)

22. sec(960°) 23. csc(-420°) 24. tan(585°) 25. cot(-495°)

Use a calculator to find the following values to three decimal places. Be sure to distinguish between degrees and radians in the angle measures.

26. cot(42°) 27. tan(76°) 28. csc(302°) 29. csc(2.11)

30. csc(7.11) 31. sec(-12°) 32. cot(103°) 33. tab(.658)

38. Give two angles (other than the ones in the table above) for which the tangent function is undefined.

39. Give three angles whose cotangent is 1.

Analysis

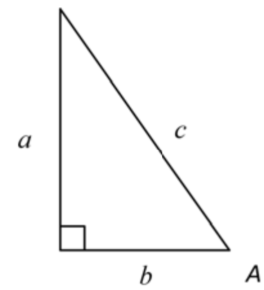
30. Show that $\tan(\theta) = \frac{1}{\cot(\theta)}$.

Section 1.5 – Practical Problems

The Big Picture

Trigonometry can be used to solve a variety of practical problems in situations like engineering and surveying because right triangles often in those fields. In this section, we're going to look at some of those situations. Looking ahead to Chapter 3, we'll eventually expand what we can do to include situations that involve non-right triangles.

"Solving Right Triangles"



"Solving a triangle" refers to starting with some of the sides and angles in a triangle and then using trigonometry to find the missing sides and angles. For now, we're only going to be able to do this with right triangles. We'll see how to do this with any triangle in Chapter 2 when we talk about the Law of Sines and the Law of Cosines.

For now, we need to fall back on the trigonometric ratios from geometry. If we have a right triangle like the one to the right then we can define the three base trigonometric functions this way:

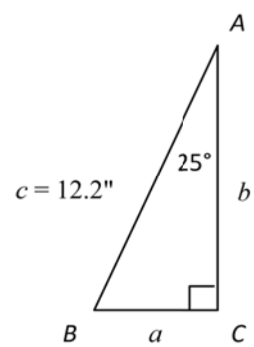
$$\sin(A) = \frac{a}{c} \qquad \cos(A) = \frac{b}{c} \qquad \tan(A) = \frac{a}{b}$$

Another way you'll sometimes see the definitions written, that's a little easier to remember, is in terms of "the side opposite angle A", "the side adjacent to angle A" and "the hypotenuse". That makes the definitions

$$\sin(A) = \frac{\text{side opposite the angle}}{\text{the hypotenuse}} \qquad \cos(A) = \frac{\text{side adjacent the angle}}{\text{the hypotenuse}} \qquad \tan(A) = \frac{\text{side opposite the angle}}{\text{side adjacent the angle}}$$

Example 1 – Solving Right Triangles

If you have a right triangle with a 25° angle whose hypotenuse is 12.2", find the lengths of the missing angles and sides.



With a problem like this, the first thing I want to do is make a drawing of the situation. The figure to the right has a right triangle with a 25° angle whose hypotenuse is 12.2". With this in hand, we can start thinking about the missing pieces.

The missing angle measure is the easiest part to find. We know that the sum of the measure of the angles in a triangle is always 180° which gives us

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

$$25^\circ + m\angle B + 90^\circ = 180^\circ$$

$$25^\circ + m\angle B + 90^\circ = 65^\circ$$

To find the missing side lengths, we're going to need to use our trigonometry ratios. The sine definition tells us that

$$\sin(A) = \frac{a}{c}$$

We know that $m\angle A = 25^\circ$ and $c = 12.2$. If we substitute those values into the formula, we get

$$\sin(25^\circ) = \frac{a}{12.2}$$
$$a = 12.2 \cdot \sin(25^\circ)$$
$$a \approx 12.2 \cdot .4226$$
$$a \approx 5.1557$$

To find the length of side b, we can do a similar calculation using the cosine formula.

$$\cos(A) = \frac{b}{c}$$
$$\cos(25°) = \frac{b}{12.2}$$
$$b = 12.2 \cdot \cos(25°)$$
$$b \approx 12.2 \cdot .9063$$
$$b \approx 11.0569$$

So now we have the measures of all three angles and all three sides:

$a \approx 1.557$"	m$\angle A$ = 25°
$b \approx 11.0569$"	m$\angle B$ = 65°
c = 12.2"	m$\angle C$ = 90°

Example 2 – Using Right Triangles

A surveyor needs to know the distance across a lake. He stands at a point on the shore and walks 50' in a line perpendicular to the line across the lake. From his new location, the angle to the point directly across the lake is 76.36°.

The diagram to the right illustrates the situation. Notice how the distance across the lake and the line walked by the surveyor make a right angle so the triangle in the diagram is a right triangle. We know the measure of angle *A* and the length of the side adjacent to it and we want the length of the side opposite it. If you look at the three trigonometric ratios, you should see that the tangent function has both the opposite side and the adjacent side:

$$\tan(A) = \frac{\text{side opposite the angle}}{\text{side adjacent the angle}}$$

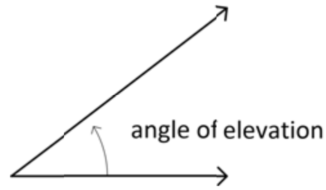
If we replace *A* with 76.36°, "side opposite the angle" with *x* and "side adjacent the angle" with 50', the equation becomes

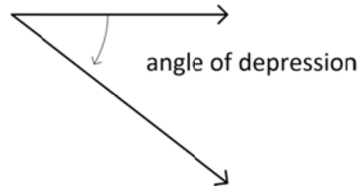
$$\tan(76.36°) = \frac{x}{50}$$
$$x = 50 \cdot \tan(76.36°)$$
$$x \approx 50 \cdot 4.1209$$
$$x \approx 206.045$$

So the distance across the lake is approximately 206.045 miles.

Angles of Elevation and Depression

Suppose you're standing in an open field and you look up at a tree or an airplane flying overhead. The angle between you and the airplane is called the "angle of elevation". Similarly, if you're standing on the edge of a canyon and looking down at a spot inside the canyon, the downward angle is called the "angle of depression". With a little trigonometry, we can use these angles to solve a variety of practical problems.





Example 3 – Altitude of an Airplane

A radar station records an airplane with an angle of elevation of 23°. Another radar station, 10 miles from the first, reports the plane as directly overhead. What's the altitude of the plane?

When you're presented with a situation like this, the first thing you should do is translate it to a picture. In this case, you have two radar stations, 10 miles apart, and a plane directly over one of them. If you draw the angle of elevation from the other station to the plane, its measure is 23°. The altitude of the plane is what we're looking for so I'll put in *x* by that distance.

Now we can start thinking about what we need to do to come up with the solution. The first thing I notice is that, because the plane is directly overhead, the triangle in the diagram is a right triangle. That means that our triangle ratios apply here. I know the length of the side adjacent to the 23° angle and I want to know the length of the side opposite it. Looking at the three ratios at the beginning of this section, the one that involves the adjacent and opposite sides is the tangent:

$$\tan(A) = \frac{\text{side opposite the angle}}{\text{side adjacent the angle}}$$

If I replace *A* with 23°, "side adjacent the angle" with 10 and "side opposite the angle" with *x*, the equation becomes

$$\begin{aligned} \tan(23°) &= \frac{x}{10} \\ x &= 10\tan(23°) \\ x &\approx 10 \cdot 0.42447 \\ x &\approx 4.2447 \end{aligned}$$

In our diagram, *x* is the altitude of the plane so we can conclude that it's approximately 4.2447 miles.

Example 4 – Height of a Tree

A person stands next to a tree and walks 20' away. From that point, the angle of elevation to the top of the tree is 48°. If the person's eyes are 5'7" above the ground, how high is the tree?

What makes this question a little unusual is that the angle of elevation is measured from the person's eyes rather than from the ground. We can apply our tangent function here but that's going to give us the height of the tree from the person's eye level:

$$\begin{aligned} \tan(48°) &= \frac{x}{20} \\ x &= 20\tan(48°) \\ x &\approx 20 \cdot 1.11 \\ x &\approx 22.2' \end{aligned}$$

Now, to get the total height of the tree we have to add on the distance from the ground up to the person's eyes. This is a point where you have to be careful with the mixed units. Remember that 22.2' isn't the same as 22' 2". In this situation, I always take the feet/inches measurements and convert them to decimal feet. For 5' 7" that means

5' 7" = 5 + 7/12 ≈ 5.58'

Now we can add the two distances together to conclude that the tree is approximately 5.58' + 22.2' = 27.78' above the ground.

When you're answering a question like this, it's always best to give the answer using the same units as the ones in the original question. The question used feet and inches so, I should convert 27.78' to that format to get the final answer. To convert .78' (the decimal part of the height) to inches, I'll multiply by 12:

$$.78 \text{ feet} = .78 \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = 9.36''$$

That makes our final answer 27' 9.36".

At this point, you might be thinking, "Why did he convert to decimal feet when, if he had thought ahead, he would have realized that he needed the final answer in feet and inches?" There are two reasons behind my decision. First, when you're doing any kind of scientific or engineering problem, the results will always be in the decimal form. Most of the practical problems that we're going to see in trigonometry fall into one of those categories so using decimals is a good habit to get into.

Second, decimal feet are a lot easier to do calculations with. Any time you do calculations with inches, in the end you have to look at the inch value and, if it's over 12", convert part of the value to feet. For example, 10' 14" would have to be rewritten as 11' 2". That's an extra step that you never have to worry about with decimal feet.

Example 5 – Height of a Cruise Ship

A person is standing on the dock of a cruise ship and wants to know how high the ship is. Unlike the previous example, he can't stand underneath the tallest part of the ship and walk away so, instead, he stands at a point where the angle of elevation is 84.3°, walks 10' further from the ship and finds the angle of elevation from that position is 82.9°. Using that information, how high is the boat from the dock?

Finding this height will require a little algebra in addition to trigonometry. The diagram to the right illustrates the situation with two variables added: *x* for the distance from the first position to the center of the boat and *y* for the height of the boat. Using those values, we can get two equations using our tangent definition:

$$\tan(84.3°) = \frac{y}{x} \qquad \tan(82.9°) = \frac{y}{x+10}$$

Here's where the algebra comes in. Notice how we know have two equations in two variables. Using some tools from algebra, we can solve those equations for both variables.

I'm going to start by solving both of the equations for *y*.

$$y = x\tan(84.3°) \qquad y = (x+10)\tan(82.9°)$$

Now I can use the Substitution Method by replacing the *y* in the second equation with *x* tan(84.3°).

$$x\tan(84.3°) = (x+10)\tan(82.9°)$$

For the next part, keep in mind that the two tangent values are both just numbers so I can treat them just like I would if I wanted to solve an equation like $3x = 4(x + 10)$. First, I'll distribute the tan(82.9°) into the parentheses.

$$x\tan(84.3°) = x\tan(82.9°) + 10\tan(82.9°)$$

Now, I'll move the *x*'s onto the same side of the equation by subtracting *x* tan(82.9°) from both sides.

$$x\tan(84.3°) - x\tan(82.9°) = 10\tan(82.9°)$$

Next, I'll factor an *x* out of the left side.

$$x(\tan(84.3°) - \tan(82.9°)) = 10\tan(82.9°)$$

Finally, I'll divide both sides by $\tan(84.3°) - \tan(82.9°)$.

$$\frac{x(\tan(84.3°) - \tan(82.9°))}{\tan(84.3°) - \tan(82.9°)} = \frac{10\tan(82.9°)}{\tan(84.3°) - \tan(82.9°)}$$

$$x = \frac{10\tan(82.9°)}{\tan(84.3°) - \tan(82.9°)}$$

That's nice because it gives us an exact answer but because this is a practical problem, we're looking for a decimal value. I can find that by using a calculator to find the two tangent values.

$$x = \frac{10\tan(82.9°)}{\tan(84.3°) - \tan(82.9°)} \approx \frac{10 \cdot 8.028}{10.019 - 8.028} \approx 40.3$$

That's the distance of the initial position from the boat but we're looking for the height of the boat (y). We can find that by substituting 40.3 for x into either of our initial equations. Using the first one gives us

$$\tan(84.3°) \approx \frac{y}{40.3}$$
$$y \approx 40.3\tan(84.3°)$$
$$y \approx 40.3 \cdot 10.019 = 403.8$$

So the height of the ship above the dock is approximately 403.8'

Having to keep track of all the trigonometric functions, e.g. the tan(82.9°) and tan(84.3°), throughout those calculations may have made the process seem complicated and messy. An alternative method is to convert the tangents to decimals right at the beginning. For example, you could go from

$$y = x\tan(84.3°) \qquad y = (x+10)\tan(82.9°)$$

to

$$y = 10.019x \qquad y = 8.028(x+10)$$

by replacing the two tangent functions with their decimal equivalents. Now the equations look a little more like the kinds of equations you would have learned to solve in an algebra class.

My preference is for the approach I used in the example. With the decimal version, you're going to have to do calculations with the decimals, e.g. multiplying and subtracting, repeatedly throughout the solution process. Keeping the tangent functions until the very end minimizes the number of decimal calculations you have to do which also minimizes the opportunities to make a mistake.

Angular Motion

Suppose you have a CD that's spinning in a CD player. Trying to determine the speed of the CD using linear units like feet per second is difficult because different points on the CD are moving at different speeds. For example, in the time it takes the CD to make one revolution, a point on the outside of the CD moves through a much greater distance than a point near the center of the CD. Because the outer point goes through a greater distance in the same amount of time, its speed has to be greater than the inner point.

The solution to this is to measure the speed in angular terms. For example, the speed of record players was measured in revolutions per minute (rpm). Thinking back to our CD example, in the time it takes the CD to make one revolution, the point on the edge and the point at the center both made one revolution so their angular speed would be the same. Here's how we define angular speed:

Angular Speed	The **average angular speed,** $\overline{\omega}$ **,** of a rotating object is equal to $$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

Talking about the "average angular speed" rather than just the "angular speed" is an important distinction in physics classes where the speed can change over time. In the situations that we're going to be discussing, the speeds will always be constant. In that special case, the two values are actually the same so, for the rest of our discussion, I'm going to put away the "bar" and just talk about the angular speed, ω.

Example 6 – Angular Speed

A record on a record player spins at 33 1/3 rpm. What's the record's speed in radians per minute?

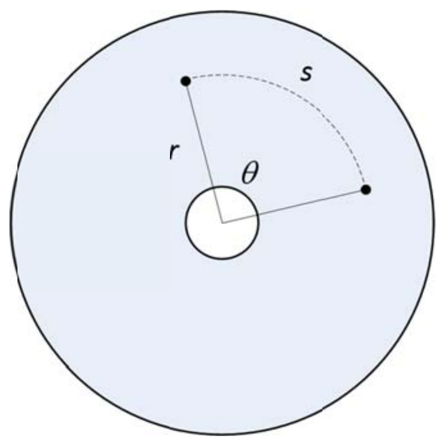
This question is asking us to do a conversion between two angular speed units, revolutions per minute and radians per minute. Like with any conversion question, the key here is knowing the conversion factor. In this case, you have to realize that 1 revolution is equal to 2π radians.

$$\frac{33\frac{1}{3}\text{ revolutions}}{1\text{ minute}} = \frac{33\frac{1}{3}\text{ revolutions}}{1\text{ minute}} \cdot \frac{2\pi\text{ radians}}{1\text{ revolution}} = 33\frac{1}{3} \cdot 2\pi\text{ radians per minute} = \left(66\frac{2}{3}\right)\pi\text{ radians per minute}$$

or approximately 209.440 radians per minute.

Angular speeds are useful because they give us a way to describe the speed of a rotating object that's constant for the entire object. There will still be times, however, when we need to know the linear speed of a point on a spinning object.

Before I start with the calculations, let's think about the situation. We have a point on a circular object like the CD to the right and we want to calculate its *linear* speed. That would be the distance, s, that the point travels divided by the time, t, that it takes to travel it. In other words,



$$v = \frac{s}{t}$$

Calculating the length of s in specific situations can be tedious so what we'd like to have is a formula that relates the (simpler to calculate) angular speed with the linear speed – hopefully one that doesn't have the s in it.

Remember that we defined radians as

$$\theta = \frac{s}{r}$$

where θ is the number of radians, r is the radius of the circle and s is the length of the arc that the angle makes in the circle. If I solve that equation for s by multiplying both sides by r, I'll get $s = r\theta$. Now, suppose I take our linear speed formula and replace the s with or $r\theta$. That would give me

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\frac{\theta}{t}$$

That brings us to the key part of the calculation. Notice that θ / t is just the angular speed of the circle, ω. If I replace θ / t with ω, the equation becomes

$$v = r\omega$$

That equation gives us a simple relationship between the linear speed of a point and the angular speed of the point: the linear speed is equal to the angular speed times the radius, i.e. the distance the point is from the center of the circle. Intuitively, this matches what we expect to happen: As the distance from the center of the circle increases, the linear speed of the point increases.

Example 7 – Angular vs. Linear Speed

A surface is spinning at a rate of 4.1 radians per minute. What is the angular speed of a point 2" from the center? What is the angular speed of a point 4" from the center?

In both parts of the question, the units are in radians so we can go directly to our angular speed equation.

r = 2"	*r* = 4"
$v = 2 \cdot 4.1 = 8.2$ inches per minute	$v = 4 \cdot 4.1 = 16.4$ inches per minute

Notice what happened there: When the radius doubled the angular speed doubled. On the one hand, that matches how we expect the linear speed to behave. When the object gets further from the center, it's covering a greater distance in the same amount of time so its linear speed should increase. On the other hand, this is also consistent with what we see in our equation. If the angular speed is a constant then $v = r\omega$ is just a linear equation so we would expect that, for example, doubling the radius is going to double the linear speed.

Example 8 – Angular vs. Linear Speed

A 33 1/3 rpm record had a diameter of 12". What was the linear speed of a point on the outer edge of the record in miles per hour?

At first glance, this might seem like a simple application of our new $v = r\omega$ formula. We know the radius and the angular speed so we can just do a little substitution and get the angular speed. There is a small detail here that you have to keep in mind when doing these calculations: When we derived the formula, the value of θ was in radians which means that the angular speed in the formula is also in radians. In practical terms, this means that, whenever you use the formula, you have to be sure that the angular speed value you use has the angle measure in radians.

For this problem, that means that we need to use the angular speed in radians per minute which we calculated in the previous example. If we substitute 6" for the radius and 209.440 for the angular speed, we get

$$v = 6 \cdot 209.440 = 1256.64 \text{ inches per minute}$$

To convert that to miles per hour we need to multiply by the appropriate conversion factors:

$$v = \frac{1256.64 \text{ inches}}{1 \text{ minute}} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \cdot \frac{60 \text{ minute}}{1 \text{ hour}} = 1.19 \text{ mph}$$

Exercises

For questions 1 – 6, find the missing angles and sides in the triangles with the given dimensions. (Remember that side *a* is always opposite ∠*A*, etc.)

1. m∠*A* = 90°, m∠*B* = 42°, *a* = 12"
2. m∠*A* = 90°, m∠*C* = 12°, *b* = 10.2'
3. m∠*A* = 18°, m∠*B* = 90°, *a* = 1 cm
4. m∠*A* = 37°, m∠*B* = 53°, *c* = 25.22"
5. m∠*C* = 90°, m∠*B* = 71.5°, *a* = 16"
6. m∠*A* = 41°, m∠*C* = 49°, *a* = 18.1"
7. A spotlight is aimed straight up on a cloudy night. An observation station 1000' from the light measures the angle of elevation of the light on the clouds at 84° from the ground. How high is the bottom of the cloud cover?

8. A tree casts a shadow that's 85' 2" long. If the angle of elevation of the shadow is 14°, how tall is the tree?
9. Century Tower at the University of Florida is a tower that holds the University's carillon.[2] An observer stands 20' from the tower and measures the angle of elevation of the top as 82.48°. If the observer is holding the measuring instrument at her eye level, 5'7" above the ground. How tall is the tower?
10. If the observer in the question 9 forgot to take the height of the instrument above the ground into account, what would be the error in her result?
11. If the observer in question 9 rounded the elevation to 82°, how would her result have changed? Assuming that the calculation in question 9 was correct, what percent error did changing the elevation by .48° introduce into the result?
12. A tree stands upright on the top of a hill. An observer stands off of the hill and measures the angle of elevation of the top of the tree at 38.3°. He moves 10' further away from the hill and measures the new angle of elevation as 33.8°. How high is the top of the tree from the base of the hill?
13. If you stand on the edge of a canyon, the angle of depression of the opposite base is 42.5°. Using a laser range finder, you determine the distance from you to the opposite base is 1061'. How deep is the canyon?
14. Suppose a tree sits at the bottom of the hill. From your position on the top of the hill, the angle of elevation of the top of the tree is 26.2° and the angle of depression to the base of the tree is 43.1°. If the straight line distance from you to the tree is 32.5', what's the height of the tree?
15. Referring to the situation in question 14, what are the straight line distances from you to the top and bottom of the tree?
16. The observation deck of a lighthouse is 45' above the ground. If an observer in the lighthouse measures the angle of depression to a boat on the water as 22.5°, how far from the lighthouse is the boat?
17. An observer on a boat measures the angle of elevation of a nearby lighthouse as 28.1°. The boat sails 30' further from the lighthouse and repeats the measurement. If the new angle of elevation is 22.77°, how tall is the lighthouse?
18. A spinning surface has a radius of 12.2". If it's spinning at a rate of 22 rpm, what is the linear velocity of a point on the outer edge? What's the linear velocity of a point that's half way to the outer edge?
19. The radius of the Earth at the equator is 6378.1 km. What's the angular speed of a point at the equator in revolutions per day? What's the angular speed in radians per day? What's the linear speed of a point on the equator in miles per hour?
20. A Ferris Wheel has a diameter of 265'. If it takes 9 minutes to make a complete rotation, what is the linear speed of a passenger in feet per minute? What's the passengers speed in miles per hour?
21. Suppose an object is on a spinning surface, 3" from the center and moving at an angular speed of 5.2 radians per second. If the object moves to a distance of 4" from the center, how fast will the object have to be spinning for the point to have the same linear speed at both locations? (You might take a look at the first question in the Analysis section before answering this one.)
22. Old style record players came in two speeds 33 1/3 rpm and 45 rpm. Calculate the angular speed of a 45 rpm record in radians per minute and degrees per minute.
23. A CD player can vary the speed of the disc so that the linear is constant. Suppose a CD is reading from a point 1" from the center and is spinning at a rate of 451 rpm. If the read head moves to 1.4" from the center, how fast

[2] A carillon is a musical instrument made up of at least 23 tuned bells usually built into some kind of tower.

will the CD have to be spinning for the linear speed at the new position to be the same as it was at the original position?

Analysis

24. Is the relationship between the linear speed (v) and the angular speed (ω) an example of direct variation or indirect variation? If the angular speed increases, what will the radius have to do in order to keep the same linear speed?