Exercises

1. A couple has four children. Draw a tree diagram that represents the possible combinations of boys and girls.
2. A computer store offers a special on computers with a choice of three different memory options, two different monitors and two different software packages. Draw a tree diagram for the sample space that shows all the possible option combinations.

3. Write a tree diagram that represents the sample space of flipping a coin four times.

Analysis

4. Look back at the BLT example. What relationship do you see between the number of each option (2 lettuces, 2 tomatoes and 3 bacons) and the total number of sandwiches? Test your relationship on the tree diagrams that you drew for Exercises 1 through 3.

Technical Writing

5. Explain the process for drawing a tree diagram. Come up with your own example to illustrate your explanation.
6. Look back at the tree diagram that I created for the BLT. Suppose you wanted to add a third kind of tomato, e.g. heirloom tomatoes. Describe how you would go about modifying the diagram to include the new option.

Section 1.2 – Two Better Methods

There's a restaurant up the street from my office that has a machine called the Coca Cola Freestyle that lets people "build their own soda". You tell it the type of soda that you want and what kind of flavoring you want and it mixes the soda you selected into a cup. You can go traditional with a cherry Coke or have something avant garde like an orange Dr. Pepper. If the machine has 13 different types of soda and 9 different flavorings, how many different drinks will it make?

This question shows the problem with our tree diagrams. The quick answer is, *a lot*. Trying to make a tree diagram would take quite a bit of time and paper. Also, when you start to make trees that big, the possibility of making a mistake somewhere in the process starts to increase and that's one of the things our tree diagrams were supposed to help prevent. If we're going to handle problems with bigger numbers like the Freestyle machine, we're going to need some better tools.

The Addition Rule

Suppose you need to go from Tampa to Louisville. You can do this by one of three different plane flights or by choosing one of two rental car options. How many ways can you make the trip? Think about this for a minute before you continue reading and see if you can come up with a number.

This brings us to our first counting rule.

The Addition Rule	If a single task can be done in one of m ways or one of n ways and the two sets of ways have no options in common then there are $n + m$ ways to do the task.

To see how you use this rule, let's walk through all of its parts and see how it applies to our travel question.

1. We have exactly one task or choice to make: picking a way to make the trip.
2. There are two groups from which we can choose: planes and rental cars.
3. None of the options in the first category (planes) is also in the second category (cars).

Because all of those requirements have been met we can use the Addition Rule to find the number of travel options. Because there are 3 options in the first group (plane flights) and 2 options in the second group (rental cars), the Addition Rule tells us that the total number of travel options is 3 + 2 = 5.

In this case, we can confirm the result by listing all of the possible options:

plane flight 1 or plane flight 2 or plane flight 3 or rental car 1 or rental car 2

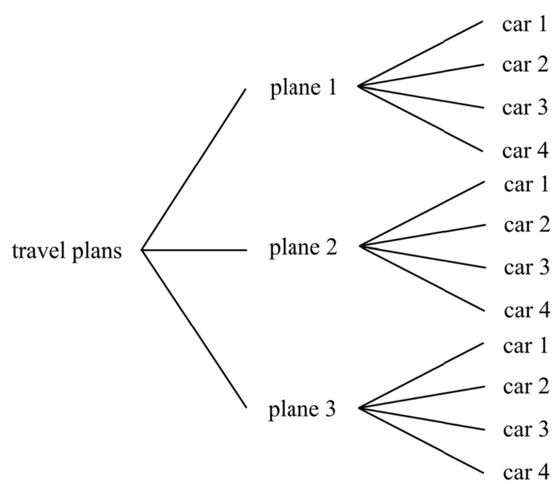
You can see from the list that the total number of travel options is five, i.e. three plane options + two car options, which corresponds with what the Addition Rule tells us.

This rule may seem a little trivial at first. While it doesn't get used as much as the Multiplication Rule that we'll talk about next, it's a tool that you should have in your back pocket. In the cases where it can be used, it makes answering questions a lot simpler.

The Multiplication Rule

Thinking about our traveler again, suppose that he decides to fly. He's still going to need a vehicle when he gets to Louisville so he'll have to rent a car. If he still has the 3 plane flights to choose from and 4 rental car choices when he lands, how many different travel combinations does he have?

We can't use the Addition Rule here because we have more than one choice to make. Our traveler has to choose both a plane *and* a rental car. To start thinking about this situation, I'm going to draw out the tree diagram. You can see what it looks like to the right. Notice how the first set of branches takes the one "travel plans" option and splits it into three, i.e. it triples the number of branches.



Now look at what the car options do. Each set of car branches takes a plane branch and splits it into four branches. This turned the original three plane options into $3 \cdot 4$ or 12 plane/car combinations.

This splitting process where every new choice multiplies the total number of branches brings us to our second counting rule.

The Multiplication Rule	If a task requires two steps, the first step can be done in *m* ways and the second step can be done in *n* ways then there are $m \cdot n$ ways that the task can be completed.

Let's think about how this applies to our traveler situation.

1. We're making one choice that requires two steps: choosing the plane and choosing a car.
2. The first choice, picking the plane, can be done in $m = 3$ ways.
3. The second choice, picking a car, can be done in $n = 4$ ways.

The Multiplication Rule tells us that we must have $m \cdot n = 3 \cdot 4 = 12$ different travel options which matches what we got from our tree diagram.

This rule tells us what to do in the case of a task with two steps, however, it works for tasks with more than two parts. No matter how many parts or choices there are, you can find the total number of ways a task can be completed by multiplying together the number of ways that its individual steps can be completed.

Example 1 – Counting Children

If a couple has three children, how many combinations of boys and girls could they have?

The way that I like to do these problems is by first drawing a line for each of the possible choices.

______	______	______
first child	second child	third child

Next, I'll put the number of options for each category on the line. In our child situation, each of the three "events" has the same two options: the child is either a boy or a girl.

2	2	2
first child	second child	third child

The Multiplication Rule tells us that the total number of possible families is equal to the product of all the individual "events" or $2 \cdot 2 \cdot 2 = 8$.

Example 2 – Calculating Addresses

Whenever a machine connects to a network, e.g. your computer connecting to the Internet, it has to be given its own unique, numeric address called an IP address. In the current version of the address system, called IP v4, every IP address is made up of four numbers, each number ranging from 0 to 255, e.g. 100.103.94.1 or 122.5.2.1. How many different IP v4 addresses are there?

We can calculate this using my "line method" from the previous example. An IP address is made up of four numbers so we have four choices to make:

______	______	______	______
first number	second number	third number	fourth number

We can put 256 different numbers from 0 to 255 in each of the four blocks.

256	256	256	256
first number	second number	third number	fourth number

If we multiply the number of options for each choice together, we get $256 \cdot 256 \cdot 256 \cdot 256 = 4,294,967,296$ different IP addresses.

Example 3 – Custom License Plates

A state allows car owners to choose their own license plate number. The number has to start with five letters and end with a single number. How many possible license plates are there?

Our custom license plates have a total of six characters (five letters plus a final number) so we're going to need to fill six blanks.

_________	_________	_________	_________	_________	_________
first letter	second letter	third letter	fourth letter	fifth letter	last digit

Each of the letter spots can have any of the 26 English letters; the numeric spot can have any of the 10 digits from 0 to 9.

26	26	26	26	26	10
first letter	second letter	third letter	fourth letter	fifth letter	last digit

Now we can use the Multiplication Rule to get the total number of license plate numbers:

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 10 = 118,813,760 \text{ numbers}$$

Example 4 – Social Security Numbers

A US Social Security Number is a ten digit number divided into three parts: a three digit number, a two digit number and a four digit number, e.g. 987-65-4320. There are a few limitations on how Social Security numbers are generated.

1. **Numbers with all 0's in any of the three groups, e.g. 000-12-1234 or 123-00-1234, aren't allowed.**
2. **Numbers with 666 or any number between 900 and 999 in the first group aren't allowed.**
3. **Numbers from 987-65-4320 to 987-65-4329 are reserved for companies to use in advertising**

Based on these rules, how many possible US Social Security Numbers are there?

Don't be put off by the list of rules. We can start this off using the line method that we've used for all of our other problems.

_________	_________	_________
group 1	group 2	group 3

The first group is a three digit number which means it has 1000 possibilities (including 000). Rule (1) tells us that we have to exclude 000 which takes the number of choices down to 999. Rule (2) excludes 666 which takes the number of choices to 998 and it excludes the 100 numbers between 900 and 999 which makes the total number of possible choices 898.

898	_________	_________
group 1	group 2	group 3

The only limitation on the second group is that it can't be 00 which means that it can be any number between 01 and 99 which gives us 99 possible choices. Similarly, the third number can be anything from 0001 to 9999 which gives it 9999 possible choices.

898	99	9999
group 1	group 2	group 3

Applying the multiplication rule gives us (so far) $898 \cdot 99 \cdot 9999 = 888,931,098$ possible numbers.

At this point, we still haven't considered rule (3). It excludes the ten numbers between 987-65-4320 and 987-65-4329. The easiest way to apply that is just to subtract out those 10 numbers from our total which gives us a final answer of $888931098 - 10 = 888,931,088$ possible Social Security Numbers.

It may not have been obvious but, on the last part of Example 4, where we applied rule (3), we used the Addition Rule. To see how, think of the Social Security numbers as being divided into two groups: advertising numbers and not advertising numbers. Because the two groups have no numbers in common, the Addition Rule tells us that the total number of Social Security numbers is

$$\text{(non-advertising numbers)} + \text{(advertising numbers)} = \text{(total number of numbers)}$$

We had already calculated that, including both types of numbers, there were 888, 931, 098 possible values and rule (3) told us that there were 10 non-advertising numbers. If we substitute those values into our Addition Rule equation, it becomes

$$\text{(non-advertising numbers)} + 10 = 888931098$$

Solving that equation for (non-advertising numbers) gives us

$$\text{(non-advertising numbers)} = 888931098 - 10 = 888931088$$

Notice how that equation is the same thing as I got from just saying, "Subtract out the numbers that we don't want."

Example 5 – Subsets of a Set

Suppose you have a set with 5 elements in it, e.g. $A = \{a, b, c, d, e\}$. How many subsets does the set have?

To answer this, we're going to have to think about the situation a little differently than in previous problems. Like the previous problems, it helps to think in terms of making choices but the key here is going to be in deciding what those choices are.

When I'm choosing a subset, I have to look at each of the five elements and answer the question, "Is this element in the subset or isn't it?" To do that, I'll start with five spaces, one for each element.

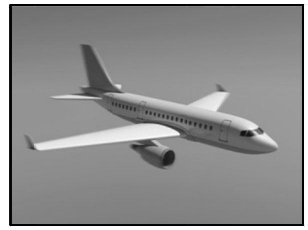
____	____	____	____	____
a	*b*	*c*	*d*	*e*

Now, for each element, I have two possible choices: Either it's in the subset or it isn't. If I put a 2 in each space, it becomes

2	2	2	2	2
a	*b*	*c*	*d*	*e*

Now, the Multiplication Rule tells us that the total number of subsets must be $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.

Exercises



1. A traveler going from Los Angeles to Denver can chose between 6 flights and 3 rental cars. He decides to return to Los Angeles by train and has 3 possible departure times. How many different travel itineraries does he have to choose from?
2. A sandwich shop offers 5 different kinds of luncheon meat, 6 different kinds of bread and 12 different toppings. If a sandwich has to have 1 kind of meat, 1 kind of bread and 3 different toppings, how many different kinds of sandwiches does the shop offer? (Assume that a topping can be chosen more than once.)
3. A license plate is made up of two parts. The first part has three letters the second part has a three digit number. How many different possible license plates are there?
4. A license plate is made up of two parts. The first part has three letters the second part has a three digit number. How many different possible license plates are there if no letter can be used more than once, e.g. ABC is okay but ABB isn't because the B is used twice?
5. How many ways can two people be assigned to ten offices? (Hint: Instead of assigning people to offices, try assigning the offices to people.)
6. In North America, a phone number is made up of three groups of digits: two three digit numbers and a fourth digit number, e.g. 123-456-7890. There are some rules to the way these numbers are assigned:

   a. The first group of numbers (the area code) can't start with a 0 or a 1.
   b. The second group of numbers (the prefix) also can't start with a 0 or a 1.
   c. The third group of numbers (the line number) can be any four digit number.

   Following these rules, how many possible North American phone numbers are there?
7. How many subsets of a set with 10 elements have more than 1 element?

8. A teenager has a music player with 85 songs on it. How many 5 song playlists can he make if he doesn't want to listen to the same song twice on a playlist?
9. A teenager has a music player with 85 songs on it. How many 5 song playlists can he make if he doesn't want to listen to the same song twice in a row?
10. A teenager has a music player with 85 songs on it. How many playlists can she make if he doesn't want to listen to the same song twice on a playlist and a playlist can have from 3 to 5 songs?
11. A school has 150 seniors, 178 juniors, 181 sophomores and 188 freshmen. If a committee is going to be made with one student from each grade, how many possible committees are there?
12. In Example 2, we saw that there are 4, 294, 967, 296 or approximately 4.3 billion possible IP addresses. That may sound like a lot but there are so many things connecting to the Internet today that the addresses are in danger of running out. The new version of the IP address scheme, IP v6, replaces IP v4's four numbers from 0 to 255 with eight numbers from 0 to 65535. How many different IP v6 addresses are there? How does this number compare to the number of IP v4 addresses?

Analysis

13. Write a statement of the Multiplication Rule that applies to a situation with four steps.
14. If a set has n elements, how many subsets does it have? I.e. find a formula for the number of subsets of a set with n elements.