# Chapter 2 – Applications

## Section 2.1 – Card Hands

In previous examples, we've used cards to come up with examples of counting problems. We can take those methods further to calculate the probability of getting specific hands in games like blackjack and poker. If you aren't familiar with calculating probabilities, I've put a quick explanation in Appendix A.

### Blackjack

To answer questions involving different card hands, it helps to focus on the specific choices that you have to make. For example, if you're dealt two cards then you would ask yourself, "How many different ways can I be dealt the first card? How many ways can I be dealt the second card?" Once you have those numbers, you can get the total number of combinations by multiplying the values together using the Multiplication Rule.

#### Example 1 – Blackjack

**In blackjack, all face cards are worth 10 points and an ace is worth either 1 point or 11. The game starts with each player being dealt two cards. If you get a hand that totals 21, i.e. an ace plus any 10 point card, then you have a blackjack and you automatically win. What's the probability of being dealt a blackjack?**

According to our probability formula:

$$P(\text{getting a blackjack}) = \frac{\text{total number of 21 point hands}}{\text{total number of 2 card hands}}$$

The only way to get 21 points is to be dealt an ace and a 10 point card so:

$$\text{total number of 21 point hands} = (\text{total ways to choose an ace}) \cdot (\text{total ways to choose a 10 point card})$$

A standard deck has 4 aces so (total ways to choose an ace) = 4 and there are 16 10 point cards (3 face cards and a 10 from each of the four suits). This means

$$\text{total number of 21 point hands} = 4 \cdot 16 = 64$$

We saw in a previous example that there are $52 \cdot 51 = 2652$ possible two card hands so our probability must be

$$P(\text{getting a blackjack}) = \frac{64}{2652} \approx .0241$$

or approximately 2.41%.

Example 2 – Blackjack

**In blackjack, after you get dealt your first two cards, you can get more cards from the dealer. Your goal is to come as close to 21 as possible without going over. Suppose you get a 9 and a 7 on your first two cards. What's the probability of your taking one more card and not going over? (Assume that no one else has been dealt any cards so that the deck, at this point, has all of the remaining 50 cards in it.)**

The current total of your hand is 16 so P(going over 21) is the same as P(getting a card greater than 5). There are nine types of cards greater than 5 (6 through king) and each of those cards comes in 4 different suits so there are a total of $5 \cdot 4 = 20$ total cards that could make you go over. Of those 20 cards, 18 of them are still in the deck. (You have to subtract out the 9 and the 7 that are already in your hand.) That gives us

$$P(\text{going over}) = P(\text{getting 6 or higher}) = \frac{\text{cards in the deck that are 6 or higher}}{\text{total cards in the deck}} = \frac{18}{50} = 0.36$$

or 36%.

## A Different Approach

With a blackjack, the rules are pretty simple and you only start off with two cards so it was easy just to count the number of cards that fell into each category. With poker hands, the situations are going to be a little more complicated. The procedure for counting those hands will usually involve two steps: first, chose a value for a card and then chose a suit for it. I'm *not* suggesting that you use this method in every situation. In Example 1, for example, it was clearly easier just to count the actual cards and say, "There are a total of 16 ten's, jack's, queen's and king's." What I want you to see here is how the method works in a simple case so that you have it in your toolbox when we get to more complicated problems.

Suppose I wanted to find (total ways to choose a ten point card). Using the method of choosing a value then choosing a suit, the process would look like this:

$$(\text{number of ten point card choices}) = (\text{ways to choose the card's value}) \cdot (\text{ways to choose the card's suit})$$

There are a total of 4 types of ten point cards. We're only thinking about the numbers at this point, not the suits, so we have to choose one of 10, jack, king or queen. That makes

$$\text{ways to choose the card's value} = \begin{pmatrix} \text{4 types of 10 point cards} \\ \text{choose 1 of them} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{4!}{1!(4-1)!} = 4$$

Now, I'll apply the same reasoning to the number of choices for the card's suit.

$$\text{ways to choose the card's suit} = \begin{pmatrix} \text{4 suits} \\ \text{choose 1 of them} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{4!}{1!(4-1)!} = 4$$

That makes the total number of ten card choices

$$\text{number of ten card choices} = (\text{value choices}) \cdot (\text{suit choices}) = 4 \cdot 4 = 16$$

This matches the result that we got in Example 1 from just counting the cards which confirms that the method is giving us correct results.

## Poker

Poker hands give us some more interesting examples because the hands are bigger and have more "winning" combinations. For this kind of problem, we'll often have to use our, "pick a value then pick a suit" strategy.

To write out a poker hand, we'll often just right out the numbers followed by a symbol for the suit, e.g. 2H would be the two of hearts and 5D would be the five of diamonds. A whole hand would be the individual cards separated by dashes so

3H-3D-3S-6H-6D would be the three of hearts, the three of diamonds, the three of spades, the six of hearts and the six of diamonds. If the suits don't matter, I'll just leave out the letters, e.g. 3-3-3-5-5 would be a hand with three 3's and two 5's of any suit.

### Example 3 – Getting a Straight Flush

**A straight flush is a hand where you have five cards in a row, all of the same suit, e.g. 2-3-4-5-6 of hearts or 6-7-8-9-10 of diamonds. How many ways are there that you can be dealt a straight flush?**

To start the process, we need to pick the first card in the straight. The value of this card can be anything from an ace up to a ten. The card can be any of the four suits which means the number of combinations for the first card is:

$$\text{choices for the first card} = \begin{pmatrix} \text{10 possible values} \\ \text{choose 1 of them} \end{pmatrix} \cdot \begin{pmatrix} \text{4 possible suits} \\ \text{choose 1 of them} \end{pmatrix} = \binom{10}{1}\binom{4}{1} = 10 \cdot 4 = 40$$

How many choices do we have for the second card? The answer to that is 1. We don't get to pick the number because it has to be 1 more than the first card we choose and we don't get to pick the suit because it has to be the same as the first card we choose. In other words, once we've chosen the first card, we don't get to make any more choices – all of the remaining cards are determined by the first choice.

That means that there are a total of 40 possible straight flushes.

So what's the probability of being dealt a straight flush? There are 52 cards in a standard deck and, to form a hand, we need to choose 5 of them. Because the order of the cards in a poker hand doesn't matter, this is an example of a combination.

$$\text{total five card hands} = \binom{52}{5} = \frac{52!}{5!47!} = 2{,}598{,}960$$

That makes the probability of being dealt a straight flush

$$P(\text{straight flush}) = \frac{40}{2598960} = 0.00001539$$

or 0.001539%.

### Example 4 – Getting a Four of a Kind

**A "four of a kind" is a hand where you have four cards with the same value. How many ways are there to be dealt four of a kind?**

Here are the choices that we need to make:

**Choice 1:** Pick the value of the first card.

**Choice 2:** Pick the value of the fifth card.

**Choice 3:** Pick the suit of the fifth card.

Once we pick the value of the first card, we know the values of the second, third and fourth cards – they have to have the same value as the first card (but with a different suit). We also don't need to pick the suit of the first card because we ultimately have to get all four suits for the card's value, i.e. if we chose a 6 then our hand has to have all four suits: 6 of hearts, 6 of clubs, 6 of spades and 6 of diamonds.

**Choice 1:** Pick any of the 13 values for the first card: $\begin{pmatrix} \text{13 possible values} \\ \text{choose 1 of them} \end{pmatrix} = \binom{13}{1} = 13$

**Choice 2:** Pick any of the remaining 12 values for the fifth card: $\binom{\text{12 possible values}}{\text{choose 1 of them}} = \binom{12}{1} = 12$

**Choice 3:** Pick any of the four suits for the fifth card: $\binom{\text{4 possible suits}}{\text{choose 1 of them}} = \binom{4}{1} = 4$

That makes our final answer:

$$\text{number of "four of a kind"} = (\text{choice 1}) \cdot (\text{choice 2}) \cdot (\text{choice 3})$$

$$\text{number of "four of a kind"} = 13 \cdot 12 \cdot 4$$

$$\text{number of "four of a kind"} = 624$$

So there are a total of 624 possible four of a kind hands.

## Example 5 – Three of a Kind

**A "three of a kind" is a hand where you have three cards with the same value. How many ways are there to be dealt three of a kind?**

This situation is very similar to the four of a kind case but with a few additions. First, we get to pick the three suits for the "three of a kind" cards and we need to make two choices for the non-"three of a kind" cards.

**Choice 1:** Pick the value of the first card. This determines the value of the next three cards.

**Choice 2:** Choose suits for the first three cards.

**Choice 3:** Pick values for the fourth and fifth cards.

**Choice 4:** Pick the suit for the fourth card.

**Choice 5:** Pick the suit for the fifth card.

Once we pick the value of the first card, we know the values of the second, third and fourth cards – they have to be the same as the first card. Unlike the four of a kind case, we do have to pick suits for those cards because we will only have three of the four possibilities.

At this point, we've picked the three matching cards. For the two non-matching cards, we can choose any of the four suits and we have 12 choices for the possible values because we have to exclude the value that we picked for the matching cards. If we included that option then we would potentially have four of a kind instead of three of a kind.

**Choice 1:** Pick any of the 13 values for the first card: $\binom{\text{13 possible values}}{\text{choose 1 of them}} = \binom{13}{1} = 13$

**Choice 2:** Pick 3 of the available 4 suits: $\binom{\text{4 possible suits}}{\text{choose 3 of them}} = \binom{4}{3} = 4$

**Choice 3:** Pick 2 of the remaining 12 values for the fourth and fifth cards: $\binom{\text{12 possible values}}{\text{choose 2 of them}} = \binom{12}{2} = 66$

**Choice 4:** Pick any of the four suits for the fourth card: $\binom{\text{4 possible suits}}{\text{choose 1 of them}} = \binom{4}{1} = 4$

**Choice 5:** Pick any of the four suits for the fifth card: $\binom{\text{4 possible suits}}{\text{choose 1 of them}} = \binom{4}{1} = 4$

That makes our final answer:

number of "three of a kind" = (choice 1) · (choice 2) · (choice 3) · (choice 4) · (choice 5)

number of "three of a kind" = 13 · 4 · 66 · 4 · 4

number of "three of a kind" = 54,912

So there are a total of 54,912 possible three of a kind hands.

## Exercises

**Questions 1 – 4 refer to blackjack hands.**

1. Suppose you've been dealt two 8's in a game of blackjack. What's the probability of going over 21 if you take a card? What's the probability of not going over 21?
2. If you've been dealt two 10's and the dealer gets an ace, what's the probability that she gets blackjack on her next card?
3. Suppose that you've been dealt a hand whose value is 20. If the dealer has an ace, what's the probability that she gets blackjack on her next card?
4. If a player can keep track of which cards have been dealt, called "counting cards", he can significantly improve his chances of winning. To help prevent this in blackjack, casinos use more than one deck of cards at once. If a game of blackjack is being played with six decks combined together instead of one, what's the probability of being dealt blackjack? What's the probability if the game is being played with seven decks?

**Questions 5 – 11 refer to poker hands.**

5. A pair is a hand with exactly two of the same card, e.g. 2-2-3-4-7. How many different ways can you be dealt a pair? What's the probability of being dealt a pair?
6. "Two pair" is a hand with exactly two pairs, e.g. 2-2-3-3-9. How many different ways can you be dealt two pair? What's the probability of being dealt two pair?
7. A "royal flush" is a hand with the 10, jack, queen, king and ace, all of the same suit. How many different ways can you be dealt a royal flush? What's the probability of being dealt a royal flush?
8. A flush is a hand where all five cards are the same suit. How many ways can you be dealt a flush? How many ways can you be dealt a flush that isn't a royal or straight flush?
9. A straight is a hand where all five cards are in sequence but they can be of any suit, e.g. 2H-3D-4H-5S-6C. How many ways can you be dealt a straight?
10. How many ways can you be dealt a straight excluding a straight flush? How does the probability of getting a straight flush compare to the probability of getting a non-straight flush?
11. A full house is a hand where you have three cards of one value and two of another value, e.g. 3-3-3-K-K. How many ways can you be dealt a full house? What's the probability of being dealt a full house?

## Analysis

12. Give a justification for using combinations in our calculations instead of permutations.