Analysis

34. Suppose you have two arithmetic sequences with the same number of terms and you add them together. Show that the new sequence is also an arithmetic sequence. What is its interval?
35. The sum of two geometric sequences is *not* a geometric sequence. Show this is true by giving a counterexample, i.e. take two geometric sequences, add their terms together and show that the sequence you get isn't a geometric sequence.
36. What's the 15^{th} term of the arithmetic sequence that starts with 10, 12.5, 15, 17.5.
37. Suppose that you get 1 penny on the first of March. Every day after that, you get twice the number of pennies that you got on the day before. How much do you get on March 31?
38. Write the first six terms of the arithmetic sequence with $a_1 = 3$ and $d = 2$ and the geometric sequence with $b_1 = 3$ and $r = 2$. Compare the results, e.g. which type of sequence grows faster?

Technical Writing

39. Explain the difference between a geometric and an arithmetic sequence.
40. Describe the process for identifying an arithmetic sequence.
41. Why does the formula for a geometric sequence have $n - 1$ in it instead of just n? (Think about how a_1 has to equal the first term of the sequence.)

Section 3.2 – Series and the Sigma Notation

Sometimes, when we're working on problems, we run into situations where we have to add together a list of things. For example, a polynomial like $x^4 + x^3 + x^2 + x + 1$ is just the sum of five monomials. Writing out long sums can get tedious so mathematicians have a short hand way to do it that's based on the Greek letter sigma (Σ). Using this notation has four parts:

ending number

$$\sum_{i=m}^{n} a_i \longleftarrow \text{formula}$$

counter starting number

The a_i is the same as it was when we were talking about sequences – a formula that tells you what the individual terms of the sum look like. To write out the sum, you would evaluate the formula for every integer starting at m and counting up to n then add the results together.

Series	If we start with a sequence, $\{a_i\}$, and add its terms together, the result is called a **series**.

Example 1 – Writing Out a Series

Write out the series $\sum_{i=1}^{4}(i+1)$.

The $i = 1$ part tells us that i is going to start at 1 and the 4 at the top tells us that it's going to go up to 4. If I evaluate $(i + 1)$ for every integer between 1 and 4 ,I get

i	$i + 1$
1	$1 + 1 = 2$
2	$2 + 1 = 3$
3	$3 + 1 = 4$
4	$4 + 1 = 5$

To get the value of the sum, all we have to do is add up the values in the right hand column:

$$\sum_{i=1}^{4}(i+1) = 2+3+4+5 = 14$$

Example 2 – Writing Out a Series

Write out the series $\sum_{i=1}^{5} x^i$.

This series is a little different from the previous one because it has an x in it. We can still use the same procedure to find the sum, the result is just going to have an x in it.

i	x^i
1	x^1
2	x^2
3	x^3
4	x^4
5	x^5

If we add those five terms together, the sum becomes:

$$\sum_{i=1}^{5} x^i = x + x^2 + x^3 + x^4 + x^5$$

Examples 1 and 2 illustrate an important fact about series that you should be aware of: Sometimes they simplify to a number and sometimes the expanded version will still have a variable in it.

Example 3 – Writing Out a Series

Write out the series $\sum_{i=2}^{4} i^2$.

Instead of writing out the table, I'm going to take the more common approach and write the series out on a single line. To help you see where the term comes from, I'm going to write the value of i underneath it.

$$\sum_{i=2}^{4} i^2 = \underset{i=2}{2^2} + \underset{i=3}{3^2} + \underset{i=4}{4^2}$$

Example 4 – Writing Out a Series

Expand the series $\sum_{i=0}^{3}(2i)x^{i+1}$.

This time, I'm going to write the result the way we would usually expand a series by just writing out the terms. Each of the terms comes from a value of i starting with 0 and ending with 3.

$$\sum_{i=0}^{3}(2i)x^{i+1} = (2\cdot 0)x^{0+1} + (2\cdot 1)x^{1+1} + (2\cdot 2)x^{2+1} + (2\cdot 3)x^{3+1}$$

Now we can get the final result by adding those values together.

$$\sum_{i=2}^{4} i^2 = 2^2 + 3^2 + 4^2 = 4 + 9 + 16 = 29$$

$$\sum_{i=0}^{3} (2i)x^{i+1} = 0x^1 + 2x^2 + 4x^3 + 6x^4$$

$$\sum_{i=0}^{3} (2i)x^{i+1} = 2x^2 + 4x^3 + 6x^4$$

Sometimes, you'll find yourself in situations where you start with a sum, e.g. 1 + 2 + 3 + 4 + 5 + 6, and you want to simplify it by writing it using the Σ notation. Doing this can significantly simplify a series but it often takes some creativity.

Example 5 – Writing a Series

Write $x + 2x^2 + 3x^3 + \ldots + 12x^{12}$ using the Σ notation.

First, notice the … in the expression. That's a shorthand way of saying, "I'm not going to write out all the terms. Just assume that they follow the same pattern." In this kind of question, identifying that pattern is the key to simplifying the expression.

The first thing I see is that the exponents on the variable start at 1 (on the x) term and increase by 1 in each term up to 12. That makes me think that part of the formula is going to be x^i. The next thing I see is that the numbers in front of the x start at 1 and increase by 1 in each term up to 12. (Think of the x as being $1x$.) This makes me think that I can get the finished formula by putting an i in front of it: ix^i.

The sum starts at 1 and ends at 12 which makes my final result

$$\sum_{i=1}^{12} ix^i$$

Example 6 – Writing a Series

Write the series 1 + 2 + 4 + 8 + 16 using the Σ notation.

To answer this kind of question, you have to bring some creativity to the table. When I look at the numbers in the series, the first thing I notice is that they're all powers of 2:

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

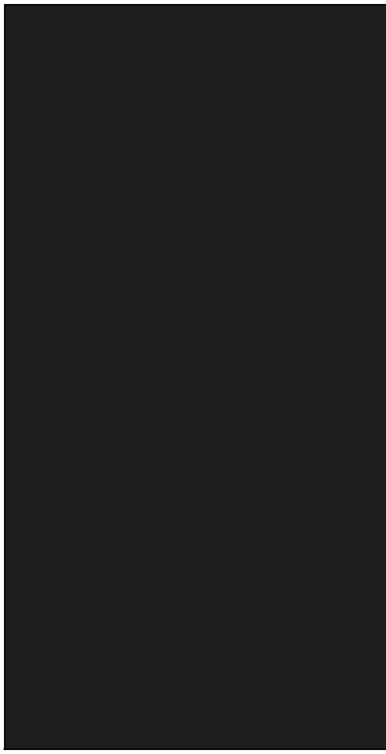
Notice how the exponents start at 0 and increase by 1 in each term. That's the behavior that the i variable has in our series. That makes me think that I can rewrite the series as

$$\sum_{i=0}^{4} 2^i$$

Techniques for Writing a Sum

Each series is unique so there's no formula that will always take you from a series to its Σ version. (It may not even be able to write a series that way, e.g. if the series is made up of random numbers.) There are a few things I always start off looking for.

1. Look for numbers that increase by 1 in every term, e.g. the exponents in Example 4 went from 1 to 2 to 3, etc. When you see that, you can replace those numbers with i in your formula.
2. Look for numbers that increase by a constant amount, e.g. 2, 5, 8, 11 starts at 2 and increases by 3 so you could write it as $(2 + i)$.



3. Look for numbers that increase by a constant exponent, e.g. 1, 3, 9, 27, 81, 243 are all powers of 3: 3^0, 3^1, 3^2, 3^3, 3^4 so you would write them as 3^i.
4. Look for numbers that increase by a constant multiple, e.g. 5, 10, 15, 20, 25 are all multiples of 5: $1 \cdot 5, 2 \cdot 5, 3 \cdot 5, 4 \cdot 5, 5 \cdot 5$. You can replace the part that increases by 1 with i which makes the formula $5i$.
5. Look for factorials. This can be tricky to see because you probably aren't as familiar with factorials as you are with multiples so, if you get stuck, and the numbers seem to be increasing pretty quickly this is a good place to go. For example, 1, 1, 2, 6 is equal to 0!, 1!, 2!, 3! so, in a formula, it would become $i!$.
6. Look for alternating signs, e.g. -1, +2, -3, +4, -5. You can write alternating signs by using $(-1)^i$. When i is an even number $(-1)^i$ is equal to +1 but when i is an odd number, it's equal to -1. That means that I could write -1, +2, -3, +4, -5 as $(-1)^i \cdot i$.

Example 7 – Writing a Series

Write $x - 2x^2 + 3x^3 - 4x^4$ using the Σ notation.

There are three things that stand out to me in this expression:

1. The exponents start at 1 and increase by 1 in each term. That makes me think there's going to be an x^i in the formula.
2. The coefficients start at 1 and increase by 1 in each term. I can add that to the formula by putting an i in front of it: ix^i.
3. The signs alternate starting with a +. This is where it gets a little tricky. If I add $(-1)^i$ to the formula, that will make the signs alternate but starting with a – where our expression starts with a +. I can fix that by changing the i to $i + 1$: $(-1)^{i+1}$. Now, when $i = 1$ that part will be $(-1)^{1+1} = (-1)^2 = +1$, when $i = 2$ it will be $(-1)^{1+2} = (-1)^3 = -1$, etc.

That makes our formula $(-1)^{i+1} \cdot ix^i$. Because we need i to go from 1 to 4, our final sum must be

$$\sum_{i=1}^{4} (-1)^{i+1} ix^i$$

Example 8 – Writing a Series

Write the series $8x^5 + 16x^6 + 32x^7 + 64x^8$.

The first thing I notice here is that the coefficients are powers of 2:

$$2^3 x^5 + 2^4 x^6 + 2^5 x^7 + 2^6 x^8$$

What makes this a little tricky is that the exponents on the 2's and the x's are out of synch with each other. One starts at 3 where the other starts at 5. We can fix that by writing the bigger numbers this way:

$$2^3 x^{3+2} + 2^4 x^{4+2} + 2^5 x^{5+2} + 2^6 x^{6+2}$$

Now, the exponents of the 2 start at 3 and go to 6 and the exponents of the x's start at 3 + 2 and go to 6 + 2. That makes our formula $2^i x^{i+2}$ where i starts at 3 and goes to 6.

$$\sum_{i=3}^{6} 2^i x^{i+2}$$

Pay close attention to what was going on in Example 7. The technique of using i for one of the parts and i + (a number) is called using an "offset". Whenever you see that two parts of the series both have the "increase by 1" pattern but they don't start at the same value, that's when you should think about using this technique. Once you've identified this situation, here's how you create the formula:

1. The smaller of the two numbers will use i. This is the exponent of the 2's in Example 7 because they are smaller than the exponents on the x's.
2. The larger of the two numbers will use i + (a number) where (a number) is the difference between the two values. In Example 7, this was the exponents on the x's. The difference between the first two exponents was $5 - 3 = 2$ so, for the x exponents, I used $i + 2$.

Example 9 – Series Formulas

Show that $\sum_{i=1}^{n} 1 = n$.

With this question, I think it will be helpful to go back to using a table to write out the terms of the sum:

i	**1**
1	1
2	1
…	
n	1

(Remember that the dots, . . ., indicate that the pattern you see in the other numbers continues. In this case, it means that there's a 1 in every column.)

What makes this series a little unusual is that there isn't an i in the formula. That's a little strange but there's nothing wrong with it. It just means that each term of the series is equal to 1, as you can see in the table. That means that

$$\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\ldots+1+1}_{n}$$

But the sum of n 1's is just n which means that $\sum_{i=1}^{n} 1 = n$

Example 10 – Series Formulas

Simplify $\sum_{i=1}^{n} x^i + \sum_{i=1}^{n} y^i$ by writing it as a single sum.

With an expression like this, it can help to write out the terms.

$$\sum_{i=1}^{n} x^i + \sum_{i=1}^{n} y^i$$

$$(x^1 + x^2 + \ldots + x^n) + (y^1 + y^2 + \ldots + y^n)$$

Now, suppose I rearrange the terms so that the ones with the same exponents are together.

$$(x^1 + y^1) + (x^2 + y^2) + \ldots (x^n + y^n)$$

Now, notice how the exponents follow our "increasing by 1" pattern. They go from 1 in $x^1 + y^1$ to 2 in $x^2 + y^2$, etc. That makes me think that I can make a new formula with $x^i + y^i$ where i goes from 1 to n. That would make the series

$$\sum_{i=1}^{n} (x^i + y^i)$$

Exercises

Write out the following series.

1. $\sum_{i=0}^{4} 3i$
2. $\sum_{i=0}^{4} x^{2i}$
3. $\sum_{i=1}^{7} (i+1)^2$
4. $\sum_{i=0}^{4} ix^{i-1}$
5. $\sum_{i=0}^{4} (-1)^i x$
6. $\sum_{i=5}^{8} \frac{i}{3}x$

7. $\sum_{i=1}^{3} \frac{i}{x}$

8. $\sum_{i=2}^{5} (x^{2i} + i)$

9. $\sum_{i=2}^{5} (-2)^i (ix+1)^2$

Write the following series using the Σ notation.

10. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$

11. $x^2 + x^4 + x^6 + x^8$

12. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$

13. $1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5$

14. $-x + 2x^2 - 4x^3 + 8x^4$

15. $-2x + 4x^2 - 8x^3 + 16x^4$

16. $-4x^2 + 5x^3 - 6x^4 + 7x^5$

17. $\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{24}+\frac{x^5}{120}$

Analysis

18. Explain why the series in example 4 is equal to $\sum_{i=1}^{3} (2i)x^{i+1}$.

19. Show that $\sum_{i=1}^{4} ax^i = a\sum_{i=1}^{4} x^i$.

20. Show that $\sum_{i=1}^{n} ax^i = a\sum_{i=1}^{n} x^i$.

21. Simplify $\sum_{i=1}^{n} x^i + \sum_{i=1}^{n} y^i$ by writing it as a single sum.

22. Show that $\sum_{i=1}^{n} (-1)^i$ is equal to 0 if n is even and is equal to -1 if n is odd.

23. Rewrite the series $\sum_{i=1}^{4} x^i$ as a new series where i goes from 2 to 5 instead of 1 to 4.

Technical Writing

24. Why do you think the series in question 22 is called a "telescoping series"?

25. Describe the process for identifying an arithmetic sequence.

26. Why does the formula for a geometric sequence have $n - 1$ in it instead of just n? (Think about how a_1 has to equal the first term of the sequence.)

Section 3.3 – Some Special Series

Arithmetic Series

Back in the late 18th century, a German school teacher assigned his room full of young students an assignment: "Add up the integers from 1 to 100." It was intended as busy work – something to keep them quiet for a little while. After just a few seconds, one of his students, young Carl Friedrich Gauss raised his hand with the answer. Ten year old Carl, who would one day become of history's most famous mathematicians had seen a pattern. If you write the numbers in order