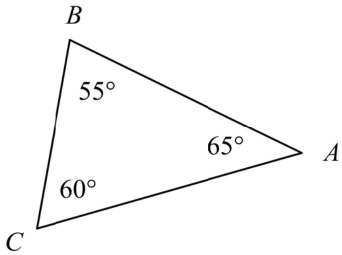
In the following triangles list the angles and sides in ascending order of size.

25.



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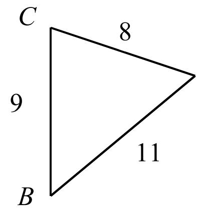

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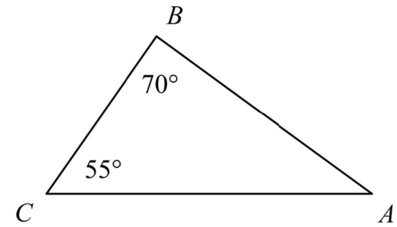


Technical Writing

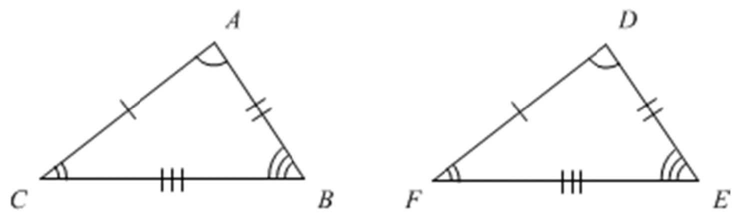
28. Discuss the relationships between the lengths of the sides of a triangle and the measures of the triangle's angles.

29. Explain why three segments of random lengths can't always be put together to make a triangle.

Section 3.3 – Congruence

Definition 3.11	Two triangles are *congruent* if and only if their corresponding angles are congruent and their corresponding sides are congruent.

This is the logical extension of our previous definitions of congruence: Two triangles are congruent, i.e. the "same", if they have the same shape (which is determined by the angles) and the same size (which is determined by the sides).



Notice how the "corresponding" part works. Each part of ΔABC "matches up" to a part of ΔDEF. For example, $\angle C$ corresponds to $\angle F$ and $\overline{AB}$ corresponds to $\overline{DE}$. How do you decide which parts go together? For purposes of deciding whether or not two triangles are congruent, you match the congruent parts. I paired $\overline{AB}$ and $\overline{DE}$ because $\overline{AB} \cong \overline{DE}$. Could I have paired, say, $\overline{DE}$ and $\overline{BC}$? Sure. But it wouldn't have helped me as far as showing the triangles are congruent because those segments aren't congruent.

When two triangles are congruent we write this just like we do with segments and angles: the names of the objects with $\cong$ in the middle. The order of the letters in the names of the triangles matters. They have to be arranged so that you can tell which parts correspond. Looking back at our example if we decide to call the first triangle ΔABC then we have to call the second one ΔDEF. Because $\overline{AB}$ is the first part of ΔABC we have to make $\overline{DE}$ the first part of the second triangle's name because $\overline{AB}$ and $\overline{DE}$ are congruent to each other.

Example 1 – Corresponding Parts

If $\Delta MNO \cong \Delta XYZ$ list the pairs of congruent angles and sides.

The angles are the easy part. You just match the first letter from *MNO* with the first letter from *XYZ*, the second letter with the second letter, etc. That gives us:

$$\angle M \cong \angle X \qquad \angle N \cong \angle Y \qquad \angle O \cong \angle Z$$

For the sides we do the same thing but we take the letters in pairs. For example, the first two letters of *MNO* and the first two letters of *XYZ* tell us that $\overline{MN} \cong \overline{XY}$. Matching the second and third letters tells us that $\overline{NO} \cong \overline{YZ}$. The last set is the one that's a little less obvious. For this pair, we have to match the first and third letters from each group. That tells us that $\overline{MO} \cong \overline{XZ}$.

Example 2 – Writing a Congruence Statement

Rewrite $\Delta MNO \cong \Delta XYZ$ as an equivalent statement with the letters in a different order.

The easiest way to do this is just to "shift" the letters. For example, I take *MNO* and shift the *M* to the end of the list I get *NOM*. Now, if I do the same thing with the other triangle, I'll get *YZX*. Since I used the same pattern on both lists, $\Delta NOM \cong \Delta YZX$ is going to be an equivalent statement. You can confirm this by finding the corresponding angles and segments from each statement and confirming that they're the same. For example, the corresponding angles from $\Delta MNO \cong \Delta XYZ$ are

$$\angle M \cong \angle X \qquad \angle N \cong \angle Y \qquad \angle O \cong \angle Z$$

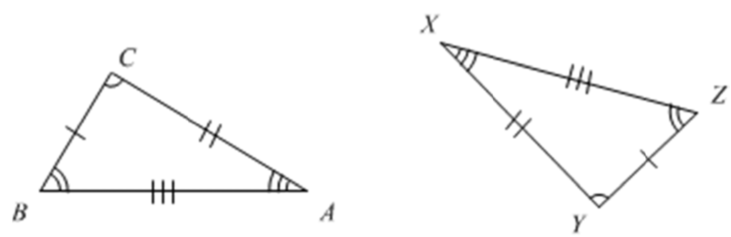
and the angles from $\Delta NOM \cong \Delta YZX$ are

$$\angle N \cong \angle Y \qquad \angle O \cong \angle Z \qquad \angle M \cong \angle X$$

The order of the angles in the lists is different but both lists pair all the same angles from the first triangle with angles in the second. If you make a list of the corresponding segments, you'll get a similar result.

Example 3 – Comparing Triangles

Write a statement that the triangles below are congruent.



The easiest way to do something like this is by looking at the angles. If we pair up the congruent angles we get $A \Leftrightarrow X$, $B \Leftrightarrow Z$ and $C \Leftrightarrow Y$ That means if we write our first triangle as ΔABC, the second triangle would have to be ΔXZY so that the pairs of corresponding angles are in the same order in both lists. That makes our final congruence statement $\Delta ABC \cong \Delta XZY$.

Proving Triangles Are Congruent

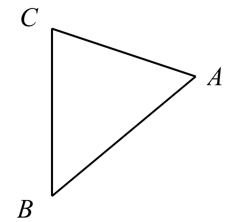
Given two pairs of triangles it's usually a chore to show that all three pairs of corresponding angles and all three pairs of corresponding sides are congruent. It turns out that it's possible to show that two triangles are congruent without showing that all six pairs of parts are. The following postulate and theorems show how we can show triangles are congruent using a smaller combination of angles and segments.

Postulate 3.1 (The SSS Postulate)	Two triangles are congruent if the three sides of one triangle are congruent to the corresponding sides of the second triangle.

There's a really simple way to confirm Postulate 3.1: Get three sticks of different lengths and arrange them in different triangles. You should see that, no matter how you arrange the sticks, you always get the same three angles in the triangle.

Theorem 3.6 (The SAS Theorem)	Two triangles are congruent if two sides and the included angle of one triangle are congruent to the corresponding sides and angle of the second triangle.
Theorem 3.7 (The ASA Theorem)	Two triangles are congruent if two angles and the included side of one triangle are congruent to the corresponding angles and side of the other.

In Theorem 3.6, "included angle" refers to the angle formed by the two sides. Looking at the triangle to the right, the included angle for sides $\overline{AB}$ and $\overline{BC}$ would be $\angle B$, the included angle for sides $\overline{AB}$ and $\overline{AC}$ would be $\angle A$ and the included angle for $\overline{AC}$ and $\overline{BC}$ would be $\angle C$.



Similarly, the "included side" refers to the side that two angles have in common. For example, the included side for $\angle A$ and $\angle B$ would be $\overline{AB}$, the included side for $\angle B$ and $\angle C$ would be $\overline{BC}$ and the included side for $\angle A$ and $\angle C$ would be $\overline{AC}$.

Example 4 – Proving Triangles Congruent

Suppose you want to show that $\Delta EFG \cong \Delta PQR$.

What would you have to show to use the SSS Postulate? What would you have to show to use the SAS Theorem?

To use the SSS Postulate, you have to show that the corresponding sides are congruent. That means we would have to show that

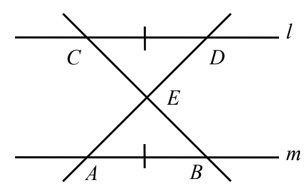
$$\overline{EF} \cong \overline{PQ} \qquad \overline{FG} \cong \overline{QR} \qquad \overline{EG} \cong \overline{PR}$$

With the SAS Theorem, we have more flexibility. We would have to pick two sides and the included angle from the first triangle, say $\overline{EF}$ and $\overline{FG}$ for the sides and $\angle F$ for the angle. Then we have to take the corresponding sides and angles from the other triangle. $\overline{PQ}$ corresponds to $\overline{EF}$, $\overline{QR}$ corresponds to $\overline{FG}$ and $\angle Q$ corresponds to $\angle F$. To use the SAS Theorem, we have to show that those corresponding parts are congruent, i.e.

$$\overline{EF} \cong \overline{PQ} \qquad \overline{FG} \cong \overline{QR} \qquad \angle F \cong \angle Q$$

Example 5 – Proving Triangles Congruent

Is $\Delta CDE \cong \Delta BAE$ if $l \parallel m$?



We know that $\overline{CD} \cong \overline{AB}$ because the diagram says so. We also know that $\angle DCE \cong \angle ABE$ and $\angle CDE \cong \angle BAE$ because they are the alternate interior angles of a pair of parallel lines. So now we've shown that two pairs of corresponding angles are congruent[14] and that the segments between them are congruent. According to the ASA Theorem this makes $\Delta CDE \cong \Delta BAE$.

In Example 4, there are three combinations you could have chosen to use the SAS Theorem. If you wanted to list all of the possibilities, you should start by picking another pair of sides from the first triangle, say $\overline{EF}$ and $\overline{EG}$, and their included angle and matching them with the corresponding sides in the second triangle then, finally, picking the third pair of sides from the first triangle, $\overline{FG}$ and $\overline{EG}$, and repeating the process with them.

[14] Be very careful here: We mean corresponding angles of the two triangles, not corresponding angles of the parallel lines.

Warning

In Example 5, we could have ignored the side and shown that all three angles are congruent. (The remaining pair are vertical angles.) Claiming that this is enough to make the triangles congruent would have been a major mistake. There is no Angle-Angle-Angle or "AAA" Theorem. To see why, draw a triangle and then draw a second one with the same angles but whose sides are twice the length of the original.

It’s very important that you use only the combinations of angles/sides that are specifically mentioned in the postulate and theorems in this section. Using anything else will quickly lead you into trouble.

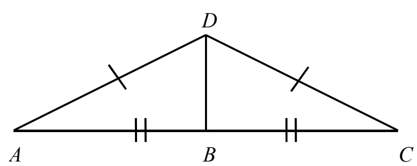
Strategies – Proving Triangles Congruent

First, think about what you're given. If you're told that two pairs of sides are congruent then you should look first at the SAS and SSS theorems because they both involve at least two pairs of sides. Similarly, if you know two angles you should look first at the ASA theorem because it's the only one that involves two angles.

When you're working these sorts of problems, or really almost any kind of geometry problem, use the picture. For this kind of problem, in particular, as soon as you convince yourself that a pair of sides or angles has to be congruent mark it that way on the diagram. Being able to physically see the relationships you know exist is easier than trying to keep it all in your head.

Example 6 – Proving Triangles Congruent

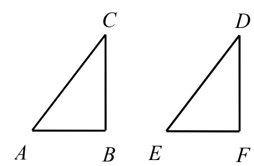
Find the congruent triangles in the figure below.



This kind of problem, where two of the triangles share a side, is pretty common. Take a look at ΔABD and ΔCBD. We know that $\overline{AD} \cong \overline{DC}$ and $\overline{AB} \cong \overline{BC}$ because they're marked that way. We also know that $\overline{DB} \cong \overline{DB}$ because a segment always has the same measure as itself. So know we've shown that each side in ΔABD is congruent to a corresponding side in ΔCBD. According to the SSS Postulate that's enough to say that $\Delta ABD \cong \Delta CBD$ even though we haven't proven anything about the angles. In fact, now that we know the triangles are congruent we can conclude that the corresponding angles are congruent.

Example 7 – Proving Triangles Congruent

In the diagram below, assume that $\overline{AB} \perp \overline{BC}$, $\overline{EF} \perp \overline{DF}$ and $\overline{AB} \cong \overline{EF}$ and both triangles are isosceles.

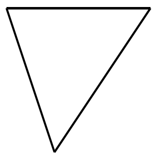


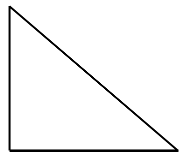
There’s a lot of information that we have to sort through here so let’s take it a piece at a time. Because $\overline{EF} \perp \overline{DF}$, we know that $\angle F$ is a right angle and, because $\overline{EF} \perp \overline{DF}$, we know that $\angle B$ is a right angle. That means that they both have the same measure, 90°, so they must be congruent, i.e. $\angle B \cong \angle F$.

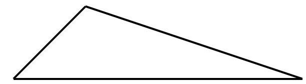
We already know that one pair of adjacent sides to angles B and F are congruent, i.e. $\overline{AB} \cong \overline{EF}$. This makes me think we can use the SAS Theorem but, to do that, we need to show that $\overline{CB} \cong \overline{DF}$. To do that, notice that $\overline{AB} \cong \overline{CB}$ and $\overline{EF} \cong \overline{DF}$ because the two triangles are both isosceles. But because $\overline{AB} \cong \overline{EF}$ that tells us that $\overline{CB} \cong \overline{DF}$ which gives us the other included sides of the right angles. Now we have two pairs of congruent sides and the angles between them so we can claim that the triangles are congruent by the SAS Theorem.

Exercises

Draw a triangle that's congruent to each of the triangles in questions 1-3. Mark the pairs of corresponding sides and angles.

1. 

2. 

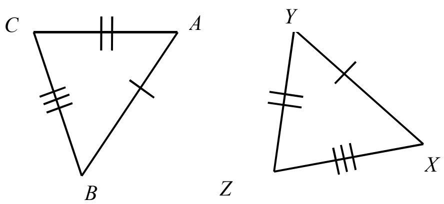
3. 

Rewrite the following statements with the letters in an order that is different but still correct.

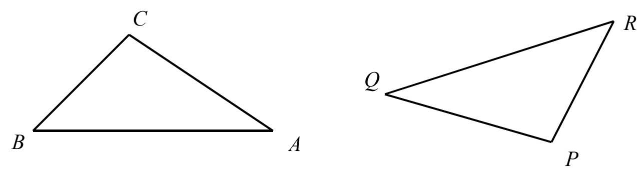
4. $\Delta ABC \cong \Delta ABC$
5. $\Delta RPG \cong \Delta MPR$
6. $\Delta PQR \cong \Delta LMN$

Assume that the triangles in questions 7 and 8 are congruent. Based on the angle/segment markings write a statement like those in questions 4-6 for each pair.

7.



8.



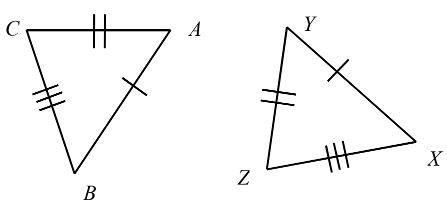
Assume that $\angle A \cong \angle Q$, $\angle B \cong \angle R$ and $\angle C \cong \angle S$.

Suppose you wanted to show that $\Delta ABC \cong \Delta MNO$.

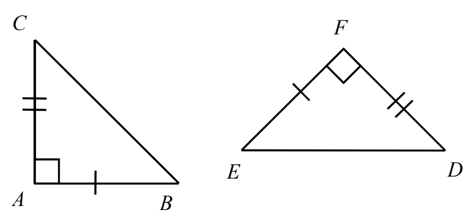
9. What congruences would you have to show to use the SSS Postulate?
10. Give one possible set of congruences that you could show to use the SAS Theorem.
11. Give one possible set of congruences that you could show to use the ASA Theorem.

The following pairs of triangles are congruent. Explain why using the SSS Postulate, the SAS Theorem or the ASA Theorem.

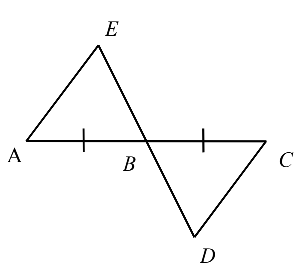
12.



13.



14. Assume that $\angle A \cong \angle C$.



15. Assume that $\overline{QP} \parallel \overline{TS}$